生命と知能の情報物理学 (1) 信号保存論理によるカオスの縁

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Information Physics on Life and Intelligence (1)
— Edge of Chaos Exhibited by Signal Conservation Logic —

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Abstract As a study on information physics this paper proposes signal conservation logic (SCL), which is a model of logic for the physical world necessarily subject to the matter conservation law. In SCL, replication, negation, and computational universality called elemental universality are all equivalent. It may mean that life under the matter conservation law eventually acquires some kind of intelligence, because intelligence has a close relation to computational universality. The matter conservation law confines the number of logical functions generated by self-organization process, and so we may be able to get rid of the difficulty caused by combinatorial explosion in the evolution of life. By applying SCL to the edge of chaos generated by cellular automata, it is proved that the square-root law is exhibited by the domino cellular automaton constructed from the SCL domino gates. This result may be regarded as a clue to the superexponential law proposed by the author.

1 Introduction

This paper investigates a certain mathematical model of the evolution of life and intelligence on the basis of Boolean logic and the matter conservation law. This theory is related with the *elemental universality*, a kind of computational universality proposed by the author [1], and can be classified as the research of natural laws in informatics [2]. The author calls such research *information physics* [3].

The elemental universality intuitively tells that, if a system based on discrete mechanism is nonlinear and negative-controlled, it is endowed with elemental universality. We can agree that life is nonlinear and negative-controlled, and that living organisms on the earth utilize DNA as their hereditary mechanism, which is digital. Thus we should think of life on the earth as having elemental universality. Another theoretical evidence that life is digital has been given by the author as a mathematical proof of the discreteness inherent to the edge of chaos [4].

The edge of chaos has been investigated as one of the central concepts in the science of complexity since S. Wolfram [5] discovered the class 4 cellular automata. Computational universality and discreteness at the edge of chaos, conjectured by Wolfram, had been the most attractive problems in this field until the author proved them [2, 3, 4].

We have still another famous problem concerning the edge of chaos. S. A. Kauffman [6, 7] proposed the square-root law observed in random Boolean networks, which is regarded as another type of the edge of chaos. The author's *superexponential law* [2, 3, 8] is also related to this square-root law. However, Wolfram's automata and Kauffman's law have not had clear mathematical connections.

The author deals with this problem on the basis of *signal conservation logic* (SCL) to be proposed here, and also proves that replication, negation, and elemental universality are all equivalent in SCL.

2 Definitions and Notations

We employ ordinary Boolean notations. Let $F : \{0,1\}^n \to \{0,1\}^n$ be a multiple-output logical element that realizes logical functions. Let the inputs of F be $X = (x_1, x_2, \cdots, x_n)$ and the outputs be $Z = (z_1, z_2, \cdots, z_n)$. We denote $z_i = f_i(x_1, x_2, \cdots, x_n)$ $(i = 1, 2, \cdots, n)$, or $Z = F(X)$, in short.

The weight, $w(V)$, of $V \in \{0,1\}^n$ is the number of 1's in V. F belongs to *signal conservation logic* (SCL), if for all inputs X its outputs $Z = F(X)$ satisfy $w(Z) = w(X)$. The notation $w(F)$ stands for $w(Z)$, and we write $F \in SCL$ or 'F is SCL,' where SCL is the set of all SCL elements. We ordinarily assume that an SCL element has delay of one unit time.

If only F is onto and one-to-one, i.e., its inverse can be defined, we say that F belongs to *information conservation logic* (ICL). The notation is $F \in ICL$ or 'F is ICL,' where ICL is the set of all ICL elements. Note that the weight condition for $w(X)$ and $w(Z)$ is not imposed on F.

The Fredkin gate [9], F: $z_1 = x_1$, $z_2 = x_1x_2 + \overline{x}_1x_3$, $z_3 = x_1x_3 + \overline{x}_1x_2$, is both SCL and ICL. This gate can realize AND, NOT, and FAN-OUT functions by applying appropriate variables and appropriate constants to its inputs: $f_2(x, y, 0) = xy$ (AND), $f_3(x, 1, 0) = \overline{x}$ (NOT), and $f_1(x, 1, 0) = x$ and $f_2(x, 1, 0) = x$ (FAN-OUT). It means that a computer can be constructed from Fredkin gates.

Given a set, S, of logical elements, if a computer (or a universal Turing machine) can be constructed from the copies of the gates in S, we say that S is *universal*. The elemental universality defined by the author [1] permits the free use of constants 0 and 1 and/or assignment of initial values to logical elements. We understand that the Fredkin gate (the set {Fredkin gate}, strictly) is universal in the sense of elemental universality. The elemental universality tells that:

Theorem 1 *A set of logical elements is universal in the sense of elemental universality if and only if it is not contained in L and P.*

Here L is the set of all linear functions, which can be written in the form $y = \alpha_0 \oplus \alpha_1 x_1 \oplus \cdots \oplus \alpha_n x_n$ for some choice of binary constants α's. The symbol $\oplus$ is exclusive OR.

P is the set of all positive functions. Binary vector $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ if and only if $x_i \leq y_i$ for all i. Logical function f is called positive if $f(x_1, \cdots, x_n) \leq f(y_1, \cdots, y_n)$ whenever $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$.

3 Basic Properties of SCL/ICL

The number, $N(n)$, of logical elements with n-inputs and n-outputs under ordinary Boolean logic is calculated as $N(n) = (2^{2^n})^n = 2^{n2^n}$. The following propositions are easily proved.

Proposition 1 *Let $N_S(n)$ be the number of possible SCL elements with n-inputs and n-outputs. Then*

$$N_S(n) = \prod_{i=0}^{n} C(n,i)^{C(n,i)}, \tag{1}$$

where $C(n,i)$ is the number of combinations of i from n objects.

Proposition 2 *Let $N_I(n)$ be the number of possible ICL elements with n-inputs and n-outputs. Then*

$$N_I(n) = 2^n!. \tag{2}$$

Proposition 3 *Let $N_{SI}(n)$ be the number of logical elements in both SCL and ICL with n-inputs and n-outputs. Then*

$$N_{SI}(n) = \prod_{i=0}^{n} C(n,i)!. \tag{3}$$

Let us consider SCL elements that can realize FAN-OUT. Here we call a FAN-OUT function 'signal replication function,' or in short 'replication.'

Lemma 1 *In SCL the realization of replication is equivalent to the realization of negation.*

Lemma 2 *If an SCL element F realizes FAN-OUT, it is universal in the sense of elemental universality.*

Theorem 2 *Replication, negation, and elemental universality are equivalent in SCL.*

Lemma 2 and Theorem 2 indicate that replication process under the matter conservation law is the 'mother' of computational universality in the sense of elemental universality. Then self-replicating organisms may be regarded as endowed with potential computational universality. Nonlinearity and nonmonotonicity of life and the self-replicating capability under the matter conservation law indicate the same digital mechanism that can acquire the capability of the universal computer.

Almost all SCL elements for large n are universal, and, in addition, the Fredkin gate is not an exceptional universal gate under the matter conservation and reversible condition.

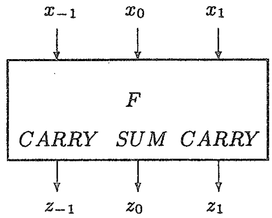


Figure 1: The domino gate.

4 Edge of Chaos by SCL

Wolfram [5] classified cellular automata into four classes. The author [4] proved the discreteness associated with the class 4 by distinguishing the class 3 from the other classes. The definition of divergence [4] tells that the width of nonzero signals and the number of nonzero signals must be both unbounded in the chaotic phase, which means that SCL cannot exhibit the class 3 phenomena under this definition if the number of cells is unbounded. (If the number of cells is finite K as in cyclically connected automata, the class 3 with periods of $O(2^K)$ can be exhibited.)

We introduce a symmetry condition to SCL elements, which we call symmetric SCL elements or, shortly, SSCL elements.

The variable names are changed from $x_1, x_2, \cdots, x_n$ to $x_{-m}, \cdots, x_0, \cdots, x_m$ and in similar manner for z_i's, where $n = 2m + 1$. We assume that $z_{-i} = z_i$ for $i = 1, \cdots, m$, although x_{-i} and x_i are independent.

We can easily prove that n must be odd for SSCL elements. It is because of the signal conservation condition. The function $z_0 = f_0(x_{-m}, \cdots, x_m)$ is uniquely determined as $f_0 = x_{-m} \oplus \cdots \oplus x_m$. The number of SSCL elements for $2m + 1$ variables is as follows:

Proposition 4 *Let $N_{SS}(2m+1)$ be the number of SSCL elements with $(2m+1)$-inputs and $(2m+1)$-outputs. Then*

$$N_{SS}(2m+1) = \prod_{i=0}^{m} C(m,i)^{C(2m+1,2i)+C(2m+1,2i+1)}. \tag{4}$$

Let us consider the only one SSCL element with three variables. It is expressed as

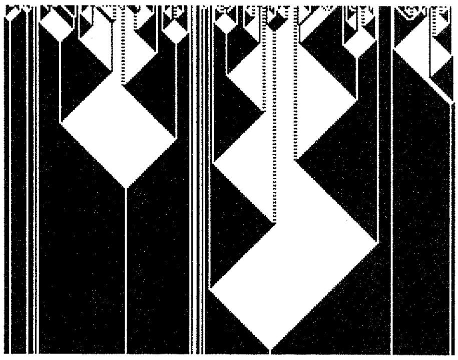
$$z_0 = x_{-1} \oplus x_0 \oplus x_1, \tag{5}$$

$$z_{-1} = z_1 = x_{-1}x_0 + x_0x_1 + x_1x_{-1}. \tag{6}$$

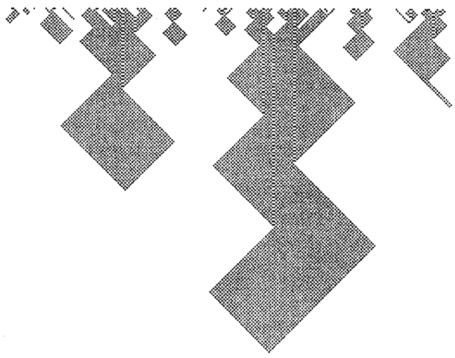
This element is also symmetric with respect to all input variables, but is realized under a very simple constraint such that $z_{-1} = z_1$. (The number of SSCL elements symmetric with respect to all input variables is $\prod_{i=0}^{m} C(m,i)^2$.)

Note that this element happens to be a one-digit full adder, with z_0 as *SUM* and z_{-1} and z_1 as *CARRY*, as depicted in Fig. 1. The author calls this element the *domino gate* due to the analysis below.

We construct a one-dimensional cellular automaton from domino gates with a unit delay. We call this automaton the *domino automaton*. The number of cells, i.e., domino gates, is unbounded in both directions. At time t, the inputs and outputs of cell F_j are expressed as $x_{j,i}^{(t)}$ and $z_{j,i}^{(t)}$. Then $x_{j,-1}^{(t)} = z_{j-1,1}^{(t)}$, $x_{j,0}^{(t)} = z_{j,0}^{(t)}$, and



(a) Outputs z_0's



(b) Outputs z_{-1}'s and z_1's

Figure 2: Edge of chaos by the domino automaton.

$x_{j,1}^{(t)} = z_{j+1,-1}^{(t)}$, and from these values $z_{j,i}^{(t+1)}$'s are calculated.

At time -1, values to the cells in a finite interval are applied such that $x_{j,0}^{(-1)} = 0$ and $x_{j,-1}^{(-1)}$ and $x_{j,1}^{(-1)}$ are random numbers 0 or 1 with equal probability 1/2. Let the left and right ends of the cells with nonzero input signals be F_L and F_R, respectively, where $R = L + N - 1$. Note that the weight $w(X)$ for each gate in this interval is 1 on an average. Other cells ($j < L$ or $R < j$) are all fed by 0's. If N is sufficiently large, the initial values of F_j ($j = L, L+1, \cdots, R$) at time 0 are $(0,0,0)$ with probability 1/4, $(0,1,0)$ with 1/2, and $(1,0,1)$ with 1/4. The values at time -1 do not have meaning other than that the initial values for F_j's at time 0 have this distribution.

An example of the state transitions by the domino automaton is shown in Fig. 2 as two figures for outputs z_0's and $z_{\pm 1}$'s, respectively, where white stands for 0 and black for 1, and time proceeds downward with the uppermost horizontal regions as the initial values at time 0. These patterns will be classified as the edge of chaos.

The pattern in Fig. 2(a) contains vertical white lines against the black background. We can prove that the number $n_0(N)$ of white lines in equilibrium is $O(\sqrt{N})$ on an average for this automaton.

Intuitively speaking, it is because the initial value assignment fluctuates with the order $O(\sqrt{N})$. If the number of 1's is fewer than that of 0's, such fluctuation results in white lines in equilibrium. We shall give an outline of the proof of this square-root law.

```
·  0  ·  ·  ·  2  ·  ·  ·  ·  0  ·
·  0  ·  ·  2  0  2  ·  ·  ·  0  ·
·  0  ·  2  0  2  0  2  ·  ·  0  ·
·  0  2  0  2  0  2  0  2  ·  0  ·
·  ·  0  2  0  2  0  2  0  2  0  ·
·  ·  ·  0  2  0  2  0  2  0  ·  ·
·  ·  ·  ·  0  2  0  2  0  ·  ·  ·
·  ·  ·  ·  ·  0  2  0  ·  ·  ·  ·
·  ·  ·  ·  ·  ·  0  ·  ·  ·  ·  ·
·  ·  ·  ·  ·  ·  0  ·  ·  ·  ·  ·
```

(a) Rhombus formation caused by s_2

```
·  0  2  0  2  0  3  0  2
·  ·  0  2  0  2  1  2  0
·  ·  ·  0  2  0  3  0  2
·  ·  ·  ·  0  2  1  2  0
·  ·  ·  ·  ·  0  3  0  2
·  ·  ·  ·  ·  ·  ·  2  0
·  ·  ·  ·  ·  ·  2  0  2
·  ·  ·  ·  ·  2  0  2  0
```

(b) Reflection by s_3

Figure 3: Analyses of rhombic patterns.

Let $s_0 = (0,0,0)$, $s_1 = (0,1,0)$, $s_2 = (1,0,1)$, and $s_3 = (1,1,1)$. These cover all the possible states of a cell. State s_0 is a quiescent state such that, if all F_j's are in s_0, the automaton is in equilibrium. We can also regard s_1 as another quiescent state, because, even if s_0 and s_1 are randomly assigned to cells, the automaton is still in equilibrium. That is, $CARRY = 0$ means the quiescent states.

We introduce the total order $s_0 < s_1 < s_2 < s_3$. The symbol $F_j^{(t)}$ represents F_j's state at time t. The weight $w(F_j)$ at time t is denoted by $w(F_j^{(t)})$, or $w(s_i)$ if $F_j^{(t)} = w(s_i)$.

The behavior of the domino automaton somewhat resembles that of the sand pile collapsing model proposed by Bak [11]. The signals in the domino automaton can be regarded as sand particles sliding down the slopes of sand piles.

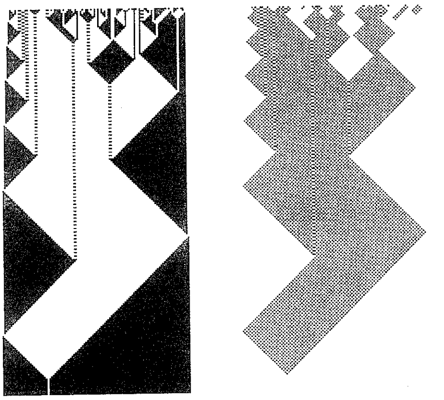
Fig. 3 depicts the state transitions for 'rhombic' patterns typical in Fig. 2. Here 0, ·, 2, and 3 represents s_0, s_1, s_2, and s_3, respectively.

In Fig. 3(a), an s_2 surrounded by s_1's generates a 'rhombus'. The effects caused by this s_2 are propagated in both directions at a speed of a cell per time step; delimited and reflected by s_0's at both sides; collide at a middle point to generate an s_0. The weight $w(s_2)$ is delivered to two s_0's at both ends, and an s_0 is generated within this interval. Fig. 3(b) depicts the signal reflection caused by s_3, whose analysis is also easy.

Such analyses tell that the excessive weights (> 1) associated with cells are gradually delivered to s_0 cells step by step, and that equilibrium is eventually attained. Since such process propagating cell by cell fairly resembles the game 'falling dominoes,' the name 'domino' is attached to this automaton. Other state transition patterns exhibited by the domino automaton have similar characteristics and easily analyzed.

Proposition 5 *The domino automaton reaches equilibrium in finite time steps.*

Let E be the average number of excessive 0's for



(a) z_0's (b) z_{-1}'s and z_1's

Figure 4: Excessive weights result in few s_0's.

$w(L,R) - N < 0$. We can approximately evaluate E using Stirling's formula.

$$\begin{aligned} E &= 2^{-2N+1}\sum_{k=0}^{N}(N-k)C(2N,k) \\ &\approx \sqrt{N/\pi}. \end{aligned} \tag{7}$$

Therefore we obtain $E \approx \sqrt{N/\pi}$, which is $O(\sqrt{N})$. It makes a good accordance with exhaustive enumeration.

Some of the excessive weights are transferred to cells outside the interval between L and R. For sufficiently large N, if $w(L,R) - N$ is $O(\sqrt{N})$ and negative, such weights are negligibly small in comparison with N, because most of locally excessive weights are absorbed by s_0's in this interval. Hence the number of s_0's between L and R in equilibrium is still $O(\sqrt{N})$.

If $w(L,R) - N$ is positive and $O(\sqrt{N})$, most of the excessive weights are moved to cells outside this interval. The number of s_0's that can remain in this interval after equilibrium is small, as in the case shown in Fig. 4. The probabilities that $w(L,R) - N$ is positive and negative are equal. Therefore, on an average, the number of s_0's that can remain in this interval of width N is still $O(\sqrt{N})$.

Kauffman observed in his random Boolean network experiments the fact that the distribution of state cycle lengths is skewed rather than bell-shaped Gaussian. Many random networks had very short state cycles. The distribution of s_0's caused by the domino automaton has a property somewhat similar to his result, although the distribution form may not be the same.

Theorem 3 *There exist cellular automata that exhibit the square-root property in equilibrium at the edge of chaos.*

5 Discussions

The computation of the numbers of various SCL elements tells that SCL may get rid of the difficulty caused by combinatorial explosion in developing probabilistic theory of evolution.

The square-root law discovered at the edge of chaos may have some relation to the superexponential law proposed by the author [2, 3, 8]. This superexponential law has another predecessor, H. Hart [13], who made an intensive investigation of the log-log law in technological innovations in 1920's – 1950's.

U. Frisch, et al. [10] and P. Bak, et al. [11] are examples of SCL.

The edge of chaos exhibited by the domino automaton can be regarded as a kind of relaxation process [3, 8]. We shall refer to this type of process as the *domino relaxation*. Such slow relaxation may find some relation to the $1/f$ power law [14]. Also see Lemma 4 by the author [4] that proves a strict $1/f$ law by graph theory.

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References

[1] Inagaki, K.: *IEICE Trans. Inf. & Syst.*, Vol. E81-D, No. 8, pp. 767–772 (1998).

[2] Inagaki, K.: *IEICE Trans. A*, Vol. J81-A, No. 9, pp. 1230–1237 (1998). English translation: *Electronics and Communications in Japan, Part III*, Scripta Technica, Vol. 82, No. 11, pp. 1–8 (1999).

[3] Inagaki, K.: *ISPJ Trans.*, Vol. 40, No. SIG2(TOM1), pp. 68-75 (1999).

[4] Inagaki, K.: *ISPJ Trans.*, Vol. 40, No. SIG2(TOM1), pp. 76–81 (1999).

[5] Wolfram, S.: *Physica D*, Vol. 10, pp. 1–35 (1984).

[6] Kauffman, S. A.: *J. Theoretical Biology*, Vol. 22, pp. 437–467 (1969).

[7] Kauffman, S. A.: *The Origins of Order*, Oxford University Press (1993).

[8] Inagaki, K., et al.: *Pattern Recognition*, Vol. 17, pp. 85–108 (1984).

[9] Fredkin, E. and Toffoli, T.: *Int. J. Theoretical Physics*, Vol. 21, Nos. 3/4, pp. 219–253 (1982).

[10] Frisch, U., et al.: *Complex Systems*, Vol. 1, pp. 649–707 (1987).

[11] Bak, P., Tang, C., and Wiesenfeld, K.: *Physical Review A*, Vol. 38, pp. 364–374 (1988).

[12] Turing, A. M.: *Philos. Trans. Roy. Soc. London*, Vol. B237, pp. 37–72 (1952).

[13] Hart, H.: *Technology and Social Change* (Allen, F. R., et al. (eds.)), Appleton-Century-Crofts, pp. 27–55 (1957).

[14] Musha, T. and Kobayashi, R.: *Noise in Physical Systems and 1/f Fluctuations* (Musha, T., et al. (eds.)), Ohmsha, pp. 559–562 (1991).