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3次元空間における三角形分割によるパンツ体の描画

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滑らかに三角形分割された曲面を得る手法について述べる。3次元空間への曲面の埋蔵は変換行列による行列演算を繰り返すことにより得られる。この変換行列とは平面上で曲面を展開し三角形分割した図をグラフとみなしたときのそのグラフ内の頂点間の隣接行列をもとにして得られる行列である。ここでは、この手法により得られる曲面の一例として低次元トポロジーの分野においてしばしば利用されるパンツ体についてその描画方法を紹介する。

Iterative method for embedding a triangulated Pair-of-Pants in 3-space *

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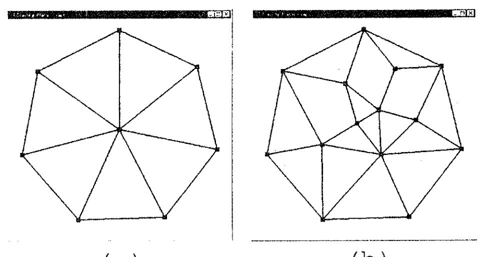
This paper describes a method for obtaining a smooth triangulated surface. A pleasing embedding of the surface in 3-space can be obtained by keeping an iteration of a *transformation matrix* derived from the adjacency matrix of the spreaded and triangulated graph of the surface on plane. As one example of the surface obtaind by our iterative method, we will present a useful surface, that we call *Pair-of-Pants* surface, in visualizing low dimentional topology.

1 Introduction

There have been many kinds of algorithms and methods for 3D surface modeling, such as subdivision surfaces[AL], NURBS surfaces, and other kinds of spline surfaces[AN]. Their method for the surfaces are based on the idea of control net and curves, which means the surfaces depend on coordinates, On the other hand, the method we present in this paper are entirely different, because the surface obtained by our method are defined only by combinatrial structure of the vertices.

The method was proposed and first used for embedding planar graphs and as the application, knot diagrams on plane *nicely-balanced* by M. Ochiai [O1]. Main operation of this method is an iteration of a canonical matrix, called *transformation matrix*, derived from the adjacency matrix of the given 3-connected simple planar graph, which has following re-



(a)　　　(b)

Figure 1: Example of planar graph; (a)initial positionings (b)converged positionings

**Key words*: transformation matrix, adjacency matrix, nicely-embedd, smooth surface, drawing torus, planer graph.

markable property. For any embedding of the graph into the plane, the boundary of the unbounded component consists of a closed, polygonal subgraph. Consider a particular placement of this subgraph as a convex polygon around the origin. Since we want a nicely-embedding of the graph, place the vertices on the convex polygon at equidistant positions on the unit circle around the origin, and all other vertices initially at the origin. Then apply the *transformation matrix* recursively to this data. The iteration quickly converges to a canonical embedding of the graph in the plane keeping the vertices on boundary of the unbounded component fixed. The embedding will be *nicely-balaned* which means intuietively that it minimizes a kind of energy for tension along the edges. Figure 1 shows an example. Figure 1(a) shows the initial positionings of vertices. The vertices you can see are fixed vertices and those which are gathering at origin. We keep the iteration unitil it converges, then figure 1(b) shows a nicely-embedding of the planar graph.

Since this method needs only the initial placement of the vertices on boundary of the unbounded component, which turns out to be fixed vertices, and an adjacency matrix of the graph, we presumed that this same method can be applied in 3-space. At first, we considered the spreaded graph of the surface that we want (see figure 3), then obtained the adjacency matrix of the graph. As the fixed vertices, we took the vertices on boundary loops in the surface. In the case of Pair-of-Pants, there are three boundary loops. We gave the initial placement of those boundary loops in 3-space, and tried the similar operation. As the result, the iteration convergs to a pleasing embedding of the surface in 3-space (figure 2).

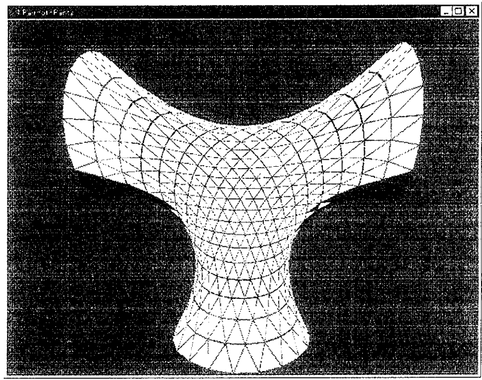


Figure 2: Pair-of-Pants surface

2 Iterative method

As we mentioned above, our method for constructing a surface is; consider a spreaded graph of the surface we want on plane, take fixed vertices among vertices of the graph and give the initial placement of them, then transform other vertices to each certain positions, which turns out the embedding of the graph will be nicely-balanced, by the iterative method. In this section, we describe the iteratation in detail.

Let G denote a graph obtained by spreading a surface on the plane, and suppose G has n vertices. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of vertices in G, and $V_O = \{v_1, v_2, \ldots, v_r\}$ be the set of fixed vertices. Let $V_I = V - V_O = \{v_{r+1}, v_{r+2}, \ldots, v_n\}$. Let $(x^k_{v_i}, y^k_{v_i}, z^k_{v_i})$ be the coordinates of v_i,which have done the iteration k times, and denote the set of coordinates of V after k-th iteration by

$$\begin{aligned} C^k_V &= \begin{pmatrix} X^k \\ Y^k \\ Z^k \end{pmatrix} \\ &= \begin{pmatrix} x^k_{v_1}, x^k_{v_2}, \cdots, x^k_{v_n} \\ y^k_{v_1}, y^k_{v_2}, \cdots, y^k_{v_n} \\ z^k_{v_1}, z^k_{v_2}, \cdots, z^k_{v_n} \end{pmatrix}. \end{aligned}$$

Then we can describe the transformation from C^k_V to C^{k+1}_V using $n \times n$ F called *transformation matrix* as follows;

$$C^{k+1}_V = F C^k_V \tag{1}$$

F is unique for G, and its i,j entry, f_{ij} is defined by

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \le i \le r \\ a_{ij}/d(v_i) & \text{if } r+1 \le i \le n \end{cases}$$

where δ_{ij} is 1 if $i = j$, and 0 if $i \ne j$, a_{ij} is i,j entry of the adjacency matrix of graph

G, and $d(v_i)$ is degree of the vertex v_i. We have known and also had the proof that this iteration by using F will surely converge, but we don't discuss in here.

When we want just to draw the surface, if we do the iteration on computer until they converge, it might be too much time consuming. Then we have the following skillful way to avoid doing calculate repeatedly. Notice that the transformation matrix F consists of 4 blocks as follows;

$$F = \begin{pmatrix} E & O \\ B & A \end{pmatrix}$$

$$\begin{cases} E; & r \times r \text{ unit matrix} \\ O; & r \times (n-r) \text{ zero matrix} \\ B; & (n-r) \times r \text{ matrix} \\ A; & (n-r) \times (n-r) \text{ matrix} \end{cases}$$

Let $C^0_{V_O} = (X^0_{V_O}, Y^0_{V_O}, Z^0_{V_O})$ be a set of coordinates of initial positions of the vertices of V_O, and $C^\infty_{V_I} = (X^\infty_{V_I}, Y^\infty_{V_I}, Z^\infty_{V_I})$ be the converged positions of V_I. Remember that the vertices of V_O are fixed through the iteration, so what we want to get is this $C^\infty_{V_I}$. We can obtain $C^\infty_{V_I}$ by the following calculation;

$$C^\infty_{V_I} \simeq (E-A)^{-1} B C^0_{V_O} \tag{2}$$

We obtained (2) as follows. Transformation from C^k_V to C^{k+1}_V can be done by (1), that is,

$$\begin{aligned} C^{k+1}_V &= \begin{pmatrix} C^{k+1}_{V_O} \\ C^{k+1}_{V_I} \end{pmatrix} \\ &= \begin{pmatrix} E & 0 \\ B & A \end{pmatrix} \begin{pmatrix} C^k_{V_O} \\ C^k_{V_I} \end{pmatrix}. \end{aligned} \tag{3}$$

Remember the vertices of V_O are fixed, that is,

$$C^{k+1}_{V_O} = C^k_{V_O} = \ldots = C^1_{V_O} = C^0_{V_O}. \tag{4}$$

Then by (3) and (4)

$$C^{k+1}_{V_I} = B C^0_{V_O} + A C^k_{V_I}. \tag{5}$$

Since if k goes to infinity, we know each entries of $C^\infty_{V_I}$ will converge, that is, we can say

$$k \to \infty, C^{k+1}_{V_I} \simeq C^k_{V_I} \simeq C^\infty_{V_I}.$$

Therefore (5) will be

$$C^\infty_{V_I} \simeq B C^0_{V_O} + A C^\infty_{V_I}.$$

Then we could have (2).

3 Embedding Pair-of-Pants

The surfaces obtained by the iterative method we discussed in previous section depend on the adjacency relations between vertices constructing the surface, and the positions of some fixed vertices. Therefore, to obtain desirable surfaces, how we define the spreaded graph of the surfaces turns out to be the most important part of whole process. In this section, we discuss an idea of the spreaded graph to obtain Pair-of-Pants surface, and the implementation it.

3.1 Spreaded graph of Pair-of-Pants

We expect a nicely-embedding of Pair-of-Pants surface in 3-space, which we means the shape is nicely-balanced and also symmetrical. Besides nicely-balanced embedding is led by the iterative method, the symmetric property is related directly to that of the spreaded graph of the surface. Then we had figure 3 as the spreaded graph for Pair-of-Pants, which we could obtain desirable Pair-of-Pants surface. Each number in the graph indicates number of each vertices (as omission partly).

Consider two hexagons which share one edge, and are triangulated inside, as in figure 3. We can specify the density of the triangulation, and as a matter of course, the smaller we triangulate, the smoother surface we can have. To construct Pair-of-Pants surface from these two hexagons, glue the left and right upper thick line to the left and right under thick line respectively. We can imagine easily that gluing of edges yield three loops, which meets the topology of Pair-of-Pants. The idea of gluing is just to assign same numbers to those vertices. In the case of figure 3, the vertices on both the left thick upper and under lines are having same number $\{1, 6, 12, 19, 27\}$, similary for the right thick line, $\{5, 11, 18, 26, 35\}$. As for the other vertices, we give numbers in order from up to under as in figure 3. Notice that this numbering is temporary one to make the adjacency matrix easily.

Next, we have to obtain the adjacency ma-

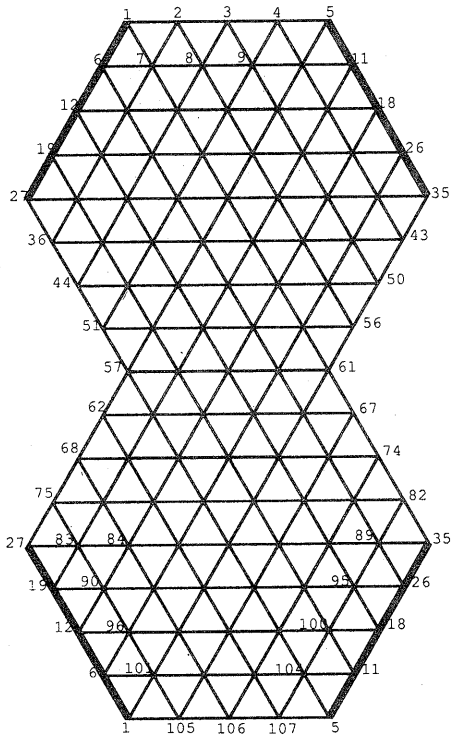


Figure 3: Spreaded graph of Pair-of-Pants

trix of this graph for the transformation matrix. Since the number of vertices have been given with regularity, we can have the adjacency matrix easily, although gluing parts are a little tricky. For example, we need to recognize that there are five vertices, that is $\{2, 6, 7, 101, 105\}$, adjacent to vertex 1.

Now, consider fixed vertices. Pair-of-Pants surface has three loops, and let L_1,L_2, and L_3 denote them. We take vertices on L_1,L_2, and L_3 as the fixed vertices. Then we have $6m$ fixed vertices, where m is the number of triangulation on an edge of a hexagon. In the case of figure 3, $L_1 = \{27, 36, 44, 51, 57, 62, 68, 75\}$, $L_2 = \{35, 43, 50, 56, 61, 67, 74, 82\}$, and $L_3 = \{1, 2, 3, 4, 5, 107, 106, 105\}$.

3.2 Implementation

Since we numbered the vertices from up to down with regularity in order to obtain the adjacency matrix easily, then before we start the iteration, we need to re-number the vertices. At first, let m be the same as above, and suppose $r = 6m$, then the vertices on three loops are

$$\begin{aligned} L_1 &= \{v_1, v_2, \ldots, v_{2m}\}, \\ L_2 &= \{v_{2m+1}, v_{2m+2}, \ldots, v_{4m}\}, and \\ L_3 &= \{v_{4m+1}, v_{4m+2}, \ldots, v_{6m}\}. \end{aligned}$$

And except for these fixed vertices, we renumber each vertices from up to down. Let n be the number of vertices in the spreaded graph. Then we have

$$\begin{aligned} V_O &= \{v_1, v_2, \ldots, v_r\} \\ &= \{v_1, \ldots, v_{2m}, v_{2m+1}, \ldots, v_{4m} \\ &\quad , v_{4m+1}, \ldots, v_{6m}\} \\ V_I &= \{v_{r+1}, v_{r+2}, \ldots, v_{n\cdot}\} \end{aligned}$$

We construct the transformation matrix for these re-numbered vertices, which can be obtain from the adjacency matrix that we had with original numbering.

Next, we specify the positionings of fixed vertices. At first, decide where we put three loops in 3-space, in precise, specify the positions of centers of the three loops, and planes where the three loops are on, and then suppose the fixed vertices are at equidistant positions on the unit circles around the centers.

Figure 2 shows the surface obtained by our method; $m = 10$, the centers and planes of L_1, L_2 and L_3 are $(0, -1, 1)$ on $y = -1$, $(0, 1, 1)$ on $y = 1$, and $(0, 0, -1)$ on $z = -1$, respectively.

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