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ベクトルコスト割当て問題の効率的解法の提案

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概要

コストをベクトル化した割当て問題について, パラメトリックな方法に基づいて準最適解を求めるアルゴリズムを提案する. 数値実験の結果, コストの２つの成分の間に強い負の相関がある場合も, 提案アルゴリズムが十分満足できるものであることが示された.

An Efficient Algorithm for Approximate Solution of the Vector Cost Assignment Problem

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Abstract

An extended model of the assignment problem is considered whose cost is a two dimensional vector. This problem belongs to the class NP-hard, and it is difficult to obtain the optimal solution. In the present paper an algorithm based on parametric analysis is presented for quasi optimal solution. Results of numerical experiments are shown, and it is concluded that the algorithm gives a satisfactory solution even when the correlation coefficient is near to −1.

1 Introduction

The assignment problem is a problem how to assign n items to n other items. Suppose you are an employer and have n employees and n jobs. Each employee can do only one job at the same time, and each job is to be done by only one employee. You have to assign each employee a job. Such a combinatorial structure is called an *assignment*, which is a bijective mapping φ between two finite sets of n elements. Since employees have different skills, you have to find the "optimal" assignment, i.e., you have to optimize an *objective function* which depends on the assignment φ. Let the *cost* (or preference) for the ith employee to do the jth job be denoted by c_{ij}, where it is assumed that smaller c_{ij}'s are preferred. If you wish to make the total cost to be the minimum, then this assignment problem is stated as follows:

P: Minimize

$$z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \dots, n),$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \dots, n),$$

$$x_{ij} \geq 0 \quad (i = 1, \dots, n;\ j = 1, \dots, n)$$

□

The variable x_{ij} takes the value 1 if the jth job is assigned to the ith employee and 0 otherwise. The problem P is a linear programming problem, and it is well known that the optimal solution of P is integer valued, i.e., 0 or 1.

Costs of the original assignment problem P are scalar valued. In practice, costs are sometimes given in a vector form. In the present paper we consider the assignment problem whose costs are given as two dimensional vector i.e., the cost for the i th employee to do the j th job is given by c_{ij} from one point of view and as c'_{ij} from another. In such an assignment problem we have two total costs, $\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$ and $\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij}$. If we are to make the larger term to be the minimum, then the problem will be stated as follows:

P$'$: Minimize

$$z$$

subject to

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} \leq z,$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij} \leq z,$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \dots, n),$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \dots, n),$$

$$x_{ij} = 0 \text{ or } 1 \quad (i = 1, \dots, n;\ j = 1, \dots, n)$$

□

Problem P$'$ is a 0-1 integer programming problem, since the unimodularity of the constraints does not hold.

In Section 2 we consider the problem P$'$ as a compound problem of the ordinary assignment problem, and define a "better" solution for this problem. In Section 3 we propose an algorithm for the problem P$'$. In Section 4 we show some numerical experiments about the problem P$'$. In the former report [1] we considered the same problem and proposed an algorithm based on binary search. In the present paper we discuss more detailed properties of the "parametric assignment problem" and propose a more efficient algorithm.

2 Parametric Assignment Problem

Since the problem P$'$ is difficult to solve, we consider a modified version of P$'$, i.e., the parametric assignment problem:

Q$'$: Minimize

$$z = t\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} + (1-t)\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \dots, n),$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \dots, n),$$

$$x_{ij} \geq 0 \quad (i = 1, \dots, n;\ j = 1, \dots, n)$$

□

If t is fixed, problem Q$'$ is an ordinary assignment problem, which is solved in time complexity $O(n^3)$.

Let us denote the optimal solution of Q$'$ by $\hat{x}_{ij}(t)$ and the value of the objective function for $\hat{x}_{ij}(t)$ by $F(t)$. Also, let us denote the solution set $\{\hat{x}_{ij}(t) \mid 0 \leq t \leq 1\}$ by S. Note that $\hat{x}_{ij}(1)$ and $\hat{x}_{ij}(0)$ are the optimal solutions of the ordinary assignment problem with costs c_{ij} and c'_{ij}, respectively. Noting that there are only finite number of distinct $\hat{x}_{ij}(t)$'s (at most $n!$ in total), we have

Lemma 1 The function $F(t)$ is piecewise linear.

Also, we can show
Lemma 2 The function $F(t)$ is concave on $0 \le t \le 1$, i.e.,

$$F(\lambda t_1 + (1-\lambda)t_2) \ge \lambda F(t_1) + (1-\lambda)F(t_2)$$

for any λ $(0 \le \lambda \le 1)$, where $0 \le t_1 \le 1$ and $0 \le t_2 \le 1$.

Let the maximum value of $F(t)$ $(0 \le t \le 1)$ be $F(t_{\text{opt}})$ $(0 \le t_{\text{opt}} \le 1)$.

Let us consider the t-$F(t)$ plane. Each segment of $F(t)$ is of the form

$$(\sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} - c'_{ij})x_{ij}(t_0))t + \sum_{j=1}^{n}\sum_{i=1}^{n}c'_{ij}x_{ij}(t_0).$$

Note that each segment of $F(t)$ corresponds to the optimal solution of the problem Q$'$ for some value of t. Let l be such a segment. The point C_1 where the extended l and the vertical line $t = 1$ meets gives the value of the objective function for the optimal assignment with cost c_{ij}. Also, the point C_2 where the extended l and the vertical line $t = 0$ meets gives the value of the objective function for the optimal assignment with cost c'_{ij}. If the gradient of l is positive, C_1 gives the value of the objective function for the problem P$'$. Similarly, if the gradient of l is negative, C_2 gives the value of the objective function for the problem P$'$. It is easy to see that when the segment l corresponds to t_{opt}, the value of the objective function for the problem P$'$ given by C_1 or C_2 is the minimum among all segments (or assignments in S). If t_{opt} where the function $F(t)$ $(0 \le t \le 1)$ takes the maximum value satisfies $0 < t_{\text{opt}} < 1$, there are more than one optimal solutions of Q$'$ at $t = t_{\text{opt}}$, namely, $\hat{x}_{ij}(t_{\text{opt}} - \Delta)$ and $\hat{x}_{ij}(t_{\text{opt}} + \Delta)$. Let us denote the better solution of the above two, i.e., the one that gives the larger value of the objective function of P$'$, by $\tilde{x}_{ij}(t_{\text{opt}})$. Thus, we have
Theorem 1 The solution $\tilde{x}_{ij}(t_{\text{opt}})$ is the best solution of S.

3 Algorithm for Finding t_{opt}

The following algorithm gives the solution $\tilde{x}_{ij}(t_{\text{opt}})$,

Algorithm

Let $t_1 = 0$ and $t_2 = 1$ as initial value ;
Solve the assignment problems Q$'$ at $t = t_1$ and at $t = t_2$;
Let $s1$ = the gradient of $\hat{x}_{ij}(t_1)$;
Let $s2$ = the gradient of $\hat{x}_{ij}(t_2)$;
if($s1 < 0$) {$\tilde{x}_{ij}(t_{\text{opt}}) = \hat{x}_{ij}(t_1)$; stop;}
if($s2 > 0$) {$\tilde{x}_{ij}(t_{\text{opt}}) = \hat{x}_{ij}(t_2)$; stop;}
Let (t_3, F') = the point of intersection of $\hat{x}_{ij}(t_1)$ and $\hat{x}_{ij}(t_2)$;
Solve the assignment problem Q$'$ at $t = t_3$;
while($F' \mathrel{!}= \hat{x}_{ij}(t_3)$){
  $s3$ = the gradient of $\hat{x}_{ij}(t_3)$;
  if($s3 == 0$){ $\tilde{x}_{ij}(t_{\text{opt}}) = \hat{x}_{ij}(t_3)$; exit; }
  if($s3 > 0$)$t_1 = t_3$;
  if($s3 < 0$)$t_2 = t_3$;
  Let(t_3, F') = the point of intersection of $\hat{x}_{ij}(t_1)$ and $\hat{x}_{ij}(t_2)$;
  Solve the assignment problem Q$'$ at $t = t_3$;
}
Let $\tilde{x}_{ij}(t_{\text{opt}})$ be the best solution of P$'$ among $\hat{x}_{ij}(t_1)$, $\hat{x}_{ij}(t_2)$ and $\hat{x}_{ij}(t_3)$;

4 Numerical Experiment

We compared the suboptimal solution $\tilde{x}_{ij}(t_{\text{opt}})$ by our proposed search algorithm with the exact optimal solution by ILOG CPLEX Version 8.0. We made comparison on the following two examples.
Example 1 Vector assignment problem whose costs c_{ij}'s and c'_{ij}'s are independently generated randomly by uniform distribution.
Example 2 Vector assignment problem whose costs c_{ij}'s and c'_{ij}'s are generated randomly by normal distribution so that c and c' are mutually correlated, positively or negatively.

4.1 Independent Random Vector Cost

In Example 1, we generated 100 instances of problem P$'$ for $n = 50, 100, 150, 200, 250, 300, 350, 400, 450, 500$. We compared our $\tilde{x}_{ij}(t_{\text{opt}})$

with the exact optimal solution by CPLEX. Let $G(\tilde{x}_{ij}(t_{\text{opt}}))$ be the value of the objective function for $\tilde{x}_{ij}(t_{\text{opt}})$, and $G(x_{ij}^{ex})$ be the value of the objective function for the exact optimal solution. We define the relative error in terms of the objective function as $(G(\tilde{x}_{ij}(t_{\text{opt}})-)G(x_{ij}^{ex})/G(x_{ij}^{ex})$.

Table 1 shows relative error, computational time by our search algorithm and by CPLEX, number of iteration, where values are the mean over 100 instances for each n.
Table 1 shows that when n grows from 50 to 500, the relative error decreases; the rate of growth of computational time is less than that by CPLEX; the number of iteration does not grow so much and looks like even constant.

4.2 Vector assignment problem with correlated costs

In Example 2 we generated 100 instances of Problem P′ for $n = 200$, where the correlation coefficient of c_{ij} and c'_{ij} varies from -1.0 to -0.2 and from 0.2 to 1.0. We compared our $\tilde{x}_{ij}(t_{\text{opt}})$ with the exact optimal solution by CPLEX. Table 2 shows the computational time, where values are again the mean over 100 instances and "independent" case was taken from Example 1.

Table 1: Mean computational time for Example 1

n	relative error (%)	iteration number	our algorithm (sec)	CPLEX (sec)
50	1.56	7.97	0.02	0.22
100	1.04	9.13	0.26	1.43
150	0.82	10.02	0.92	4.54
200	0.68	10.16	2.14	11.27
250	0.51	10.55	4.28	24.55
300	0.57	10.72	7.36	42.00
350	0.47	11.04	11.74	65.57
400	0.42	11.39	17.93	94.96
450	0.43	11.55	25.71	129.07
500	0.42	11.95	36.43	172.48

Table 2: Mean computational time for Example 2

correlation coefficient r	our algorithm (sec)	CPLEX (sec)
$-1.0 < r \leq -0.8$	8.43	217.49
$-0.8 < r \leq -0.6$	7.88	60.16
$-0.6 < r \leq -0.4$	8.01	63.49
$-0.4 < r \leq -0.2$	7.85	42.30
independent	7.34	42.00
$0.2 \leq r < 0.4$	3.38	16.00
$0.4 \leq r < 0.6$	5.51	21.15
$0.6 \leq r < 0.8$	3.98	15.63
$0.8 \leq r < 1.0$	2.50	7.79

Table 2 shows that when two costs are positively correlated, our algorithm and CPLEX give solutions quicker than when two costs are independent. In our algorithm the relative error was less than 0.5 %. The difference of CPLEX and our algorithm is more impressive when the coefficients are negatively correlated.

5 Conclusions

In the present paper we extended the classical assignment problem to an assignment problem with vector cost and proposed an algorithm based on parametric analysis. We compared the solution given by our algorithm with the exact optimal solution and concluded that our algorithm gives quasi optimal solution. Our algorithm is of polynomial time complexity. It is our conjecture that even when the cost vector is of dimension more than two, similar polynomial time algorithm that gives quasi optimal solution will exist. Detailed analysis and development of such algorithms are left for further research.

References

[1] Sakakibara, S., Y. Kamura, and M. Nakamori, Variants of Assignment Problem - Worst Cost Minimization and Vector Cost Assignment, *Proc. PDPTA 2004*, **1**, 305-310 (2004).