

[Invited Lecture]

The Maximum Clique Problem and Its Applications

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Abstract. We show some recent results in the Advanced Algorithms Research Laboratory at the University of Electro-Communications on the maximum clique problem and its applications. These are parts of our various work in the AARL at UEC with which the author is involved together with his colleagues and his students.

1 The Maximum Clique Problem

Many problems can be formulated as graphs where a graph consists of a set of vertices and a set of edges, in which the vertices stand for objects in question and the edges stand for some relations among the objects. A *clique* is a subgraph in which all vertices are pairwise adjacent[6]. Hence, a clique represents a set of objects in which every pair is related. In addition, a maximum clique of the direct product of two graphs represents a maximum matching in the two graphs[6]. Therefore, a maximum clique and maximal cliques play an important role and have received considerable attention[11, 6].

However, the so called *maximum clique problem* is considered to be very hard to solve, that is, it is proved to be an *NP-complete* problem. Nevertheless, many researchers including the author are engaged in devising as fast algorithms as possible for finding a maximum clique and generating all maximal cliques, because of their importance in practice.

1.1 Fast Algorithms for Finding a Maximum Clique

Recently, we presented a simple and fast branch-and-bound algorithm MCQ[28] for finding a maximum clique, and we improved it to get a new algorithm MCR[30], primarily by introducing more appropriate sorting of vertices at the beginning. MCR in turn was improved to a more efficient algorithm MCR-Re[31], by employing sophisticated approximate coloring and sorting of vertices. In addition, we have improved MCR-Re to have an algorithm MCS (previously called MCS₀) by localizing the memory usage in order to make more effective use of cache memory[22].

A comparison of MCQ, MCR, and MCS is shown in Table 1, where the branches correspond to the extent of search spaces[28, 30] and ω is the size of a max-

imum clique in a given graph. Some computational time comparisons with other algorithms are shown in Tables 2 and 3. In Table 2, n is the number of vertices and p the edge probability. The computer used in these experiments is a Pentium4 3.6GHz CPU operating on Linux[31]. It is confirmed that MCS is by far the fastest among all the presently existing algorithms for almost all cases.

We have also developed algorithms for weighted graphs[27, 24].

1.2 Algorithms for Generating Maximal Cliques

In addition to finding a maximum clique, generating all maximal cliques is required in many diverse applications such as clustering, data mining and others[6, 13]. We proposed an algorithm CLIQUES[29] for generating all maximal cliques.

A part of its computational time comparisons with other algorithms is shown in Table 4, where #cliques is the number of maximal cliques. The computer used in this experiment is a Pentium4 2.2GHz CPU operating on Linux[29]. CLIQUES is very fast and space efficient.

Some variations of CLIQUES have also been developed[14, 15].

1.3 Theoretical Analyses

We proved that the worst-case time complexity of CLIQUES is $O(3^{n/3}) = O(2^{0.528n})$ for an n -vertex graph, and that is *optimal* with respect to n .

Steady improvements have been made to the time complexity for finding a maximum clique in an n -vertex graph in polynomial-space from $O(2^{0.333n})$ [26] to $O(2^{0.288n})$ [9] in the last almost 30 years. We have remarkably improved this complexity to $O(2^{0.19669n})$ [16] by an algorithm that is based on CLIQUES[29, 20]. Our algorithm is also fast in practice[20, 25]. Further theoretical analysis is in progress.

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Table 1: Comparison of MCQ, MCR, and MCS

Graph		CPU time [sec]			branches $\times 10^{-3}$		
Name	ω	MCQ	MCR	MCS	MCQ	MCR	MCS
brock400.2	29	748	742	297	116,224	116,328	33,513
brock400.4	33	680	639	248	118,855	114,925	30,855
MANN_a27	126	2.61	2.54	0.78	38	38	9
MANN_a45	345	2,775	3,090	281	2,852	2,952	225
p_hat300-3	36	17	11	3	2,473	1,546	235
p_hat500-3	50	2,895	1,788	150	237,077	138,300	7,923
p_hat700-3	62	122,264	68,187	2,392	7,046,183	3,733,665	88,168
p_hat1000-2	46	2,764	2,434	221	221,797	197,147	12,618
san200.0.9.3	44	10.59	0.16	0.06	1,182	22	6
san400.0.9.1	100	32.8	3.4	0.1	708	74	2
sanr200.0.9	42	322	289	41	42,865	40,470	3,471

Table 2: CPU time [sec] for random graphs

Graph			dfmax	MCS	New	COCR
n	p	ω	[11]	(Ours)	[18]	[19]
100	0.6	11-13	0.0041	0.0016	0.0022	0.092
	0.7	14-16	0.018	0.0036	0.0067	0.12
	0.8	19-21	0.14	● 0.0078	0.065	0.15
	0.9	29-32	3.67	★ 0.013	0.66	0.20
	0.95	39-48	23.74	★ 0.0028	0.20	
	0.98	56-68	26.54	★★★ 0.00087		
150	0.7	16-18	0.36	● 0.047		0.33
	0.8	23	6.88	○ 0.23		0.75
	0.9	36-39	1,058.96	1.01		1.16
	0.95	50-59	37,436.79	★★★ 0.35		
	0.98	73-85	$> 10^5$	★★★ 0.0061		
200	0.5	11-12	0.038	0.015	0.020	0.25
	0.6	14	0.29	○ 0.072	0.17	0.52
	0.7	18-19	3.85	○ 0.41	3.02	1.65
	0.8	24-27	192.68	4.48	147.29	8.69
	0.9	40-44	$> 10^5$	73.62		○ 36.79
	0.95	58-66	$> 10^5$	★★★ 58.83		
	0.98	90-103	$> 10^5$	★★★ 0.21		
300	0.5	12-13	0.36	0.13	0.20	1.13
	0.6	15-16	4.88	○ 0.99	3.50	4.98
	0.7	19-21	144.11	★ 12.00	121.02	
	0.8	28-29	26,235.96	★ 393.57		
	0.9	49	$> 10^5$	79,628.80		
500	0.5	13-14	8.99	○ 2.79	7.25	17.43
	0.6	17	242.29	○ 40.70	183.28	
	0.7	22-23	24,998.42	★ 1,538.74		
	0.75	26	$> 10^5$	○ 20,403.68		
1,000	0.3	9-10	1.98	1.15	1.64	
	0.4	12	33.28	13.25	23.19	
	0.5	15	1,107.70	○ 290.03		
	0.6	20	$> 10^5$	● 13,554.05		
5,000	0.1	7	6.29	3.32		
10,000	0.1	7-8	137.05	○ 59.55		
15,000	0.1	8	792.57	○ 326.78		

Entries indicated by ★★★, ★, ●, and ○ represent those that are more than or equal to 1000, 10, 5, and 2 times faster than all the others confirmed within the time limits in the same row, respectively.

Table 3: CPU time [sec] for DIMACS benchmark graphs

Graph		dfmax [11]	MCS (Ours)	New [18]	χ + DF [8]	COCR [19]	MIPO [5]	Target/5 [21]
Name	ω							
brock200.1	21	14.53	★ 0.86	12.12	68.70			69.80
brock200.4	17	0.90	★ 0.14	0.22	6.04	0.91		3.60
brock400.1	27	22,051	★ 693		>10,640			>4,320
brock400.2	29	13,519	★ 297		>10,640	>415		>4,320
brock400.3	31	14,795	★ 468		>10,640			>4,320
brock400.4	33	10,633	★ 248		>10,640	>415		>4,320
brock800.1	23	> 10 ⁵	★ 9,347		>10,640			
brock800.2	24	> 10 ⁵	★ 8,368		>10,640	>415		
brock800.3	25	91,031	★ 5,755		>10,640			
brock800.4	26	78,737	★ 3,997		>10,640	>415		
c-fat500-10	126	> 10 ⁵	0.026	0.016	0.015			
hamming8-4	16	1.85	0.20	0.19	4.51	1.00	29.13	
hamming10-2	512	> 10 ⁵	○ 0.19	0.56	3.81			
johnson16-2-4	8	0.75	0.14	0.060	5.88		★ 0.0017	
MANN_a27	126	> 10 ⁵	○ 0.78	>2,232	7,647	2.75		
MANN_a45	345	> 10 ⁵	★★ 281		>10,640			
p_hat300-2	25	0.63	★ 0.018	0.22	2.23	0.61		
p_hat300-3	36	780	○ 2.55		633	5.39		
p_hat500-1	9	0.051	0.030	0.065	0.44			0.20
p_hat500-2	36	133	★★ 0.74	95.71	151			>4,320
p_hat500-3	50	> 10 ⁵	★★ 150		>10,640			>4,320
p_hat700-1	11	0.20	0.10	0.15	1.98	2.74		1.40
p_hat700-2	44	5,300	○ 5.60		1,542	25.44		>4,320
p_hat700-3	62	> 10 ⁵	★ 2,392		>10,640	>415		>4,320
p_hat1000-1	10	1.05	○ 0.49	1.30	12.14			
p_hat1000-2	46	> 10 ⁵	★★ 221		>10,640			
san200_0.9.1	70	> 10 ⁵	0.22	0.060	46.27		0.050	4.60
san200_0.9.2	60	> 10 ⁵	0.41	0.96	1,427		○ 0.15	65.60
san200_0.9.3	44	42,643	★★ 0.063		144		15.15	>4,320
san400_0.5.1	13	433	0.020	○ 0.0067	4.98		85.44	0.80
san400_0.7.1	40	> 10 ⁵	★ 0.54	>2,232	315			24.40
san400_0.7.2	30	> 10 ⁵	★★ 0.13	113	118		505	113.20
san400_0.7.3	22	> 10 ⁵	★★ 1.44		456			>4,320
san400_0.9.1	100	> 10 ⁵	★★★ 0.12		5,335			>4,320
san1000	15	> 10 ⁵	2.17	★ 0.11	2,249			
sanr200_0.9	42	86,954	★★★ 41		>10,640			
sanr400_0.5	13	2.12	○ 0.72	1.48	17.06			
gen200_p0.9.44	44	48,262	○ 0.47			1.88	13.01	
gen200_p0.9.55	55	9,281	1.23			0.96	● 0.19	
gen400_p0.9.55	55	> 10 ⁶	★ 58,502					
C125.9	34	50.05	● 0.060			0.56	46.6	
C250.9	44	> 10 ⁶	★★ 3,257					

Entries indicated by ★★★, ★★, ★, ●, and ○ represent those that are more than or equal to 1000, 100, 10, 5, and 2 times faster than all the others confirmed within the time limits in the same row, respectively.

Table 4: CPU time [sec] for random graphs

Graph		CLIQUES [29]	AMC [13]	AMC* [13]
Name	#cliques			
r1000.1	118,325	0.2	143.1	19.4
r1000.2	1,183,584	2	4,486	830
r3000.1	2,945,211	11	>86,400	5,905
r5000.1	18,483,855	87	>86,400	>86,400

2 Applications

The above algorithms and their extensions are being successfully applied to many problems. These include the followings:

- Clustering[33],
- Bioinformatics[2], [3], [4], [7], [1],
- Image processing[10],
- Design of quantum circuits[17],
- Design of DNA and RNA sequences for biomolecular computation[12], [23].

Parallel processing is also under study[32] in order to solve large practical problems.

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