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# プリソート列の特徴付けについて

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整数列がどの程度ソートされているかということを示す測度として本論文では (k,p)-sortedness という概念を導入し、整数列が (k,p)-sorted であるための必要十分条件を求めた。必要十分条件は、整数列の区間部分列による局所的な条件と、帰納的な条件とを与えている。さらに、整数列の接続行列を定義し、整数列が (k,p)-sorted であるときの接続行列の形態をｋとｐをパラメータとして記述している。

# On characterizations of presorted sequences

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We introduce the concept of $(k,p)$-sortedness as a measure of presortedness of integer sequences. It is a natural extension of $k$-sortedness introduced by Igarashi and Wood. Two types of necessary and sufficient condition for the $(k,p)$-sortedness are given. One is a local characterization and the other is a recursive characterization. We also describe a matrix representation of the $(k,p)$-sortedness.

# 1. Introduction

In some applications of sorting, sequences may roughly or nearly sorted in some sense. In [4], Igarashi and Wood defined the roughly sortedness of sequences of integers. A sequence of integers $\alpha = (a_1, a_2, \cdots, a_n)$ is *k-sorted* if $j + k < i$ implies $a_j \leq a_i$, where $k$ is a nonnegative integer. There are several other measures of nearly sortedness[3,6,7], but those are different from roughly sortedness. In [6], Mannila studied several measures of presortedness and optimal sorting algorithms for such presorted sequences. In this paper, we introduce a presortedness measure called $(k,p)$-sortedness which is an extension of the $k$-sortedness, and give some characterizations of $(k,p)$-sorted sequences. Then, we derive some basic properties. Our characterizations are classified into two categories, one is a local characterization and the other is a recursive characterization. In the local characterization, we are mainly concerned with the block size for a $(k,p)$-sorted sequence.

**Definition 1.1.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ of integers is *(k,p)-sorted* if for any $i, j$ such that $j + k < i$, $a_j + p \leq a_i$, where $k$ and $p$ are nonnegative integers.

**Definition 1.2.** Let $\alpha = (a_1, a_2, \cdots, a_n)$ be a sequence of integers. Let $V$ be the set of values in $\alpha$. The *incidence matrix* $M(\alpha)$ of $\alpha$ is the matrix whose rows correspond to positions in $\alpha$ and columns correspond to values in $V$ in increasing order, and $(i,j)$ element of $M(\alpha)$ is 1 if $a_i$ is the $j$-th value of $V$ and otherwise 0.

Example 1.3. $\alpha = (3, 5, 1, 7, 2)$

$$M(\alpha) = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 5 & 7 \\ \hline 1 & & & 1 & & \\ 2 & & & & 1 & \\ 3 & 1 & & & & \\ 4 & & & & & 1 \\ 5 & & 1 & & & \\ \hline \end{array}$$

Fig. 1.1

Example 1.4. $\alpha = (1, 3, 2, 3, 3, 1)$

$$M(\alpha) = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 1 & & \\ 2 & & & 1 \\ 3 & & 1 & \\ 4 & & & 1 \\ 5 & & & 1 \\ 6 & 1 & & \\ \hline \end{array}$$

Fig. 1.2

When $p = 0$, the $(k,0)$-sortedness is equivalent to the $k$-sortedness in [4], and therefore the $(k,p)$-sortedness is an extension of the $k$-sortedness.

**Proposition 1.5.** For $p > p'$, if $\alpha$ is $(k,p)$-sorted, then $\alpha$ is $(k,p')$-sorted. For $k > k'$, if $\alpha$ is $(k',p)$-sorted, then $\alpha$ is $(k,p)$-sorted.

**Definition 1.6.** Given a sequence $\alpha = (a_1, a_2, \cdots, a_n)$, a nonnegative integer $b$, and an integer $i$, $1 \leq i \leq n - b + 1$, the *b-block of* $\alpha$ *at position* $i$ is the subsequence $(a_i, \cdots, a_{i+b-1})$ of $\alpha$. A *b-block of* $\alpha$ is a $b$-block at some position $i$.

# 2. Characterizations of (k,p)-sorted sequences

## 2.1. Local characterization

In [4], a local characterization of the $k$-sortedness is given and it says that a sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $k$-sorted if and only if every $(2k+2)$-block is $k$-sorted.

This characterization can be extended to be a local characterization of the $(k,p)$-sortedness. We obtain the following property.

**Proposition 2.1.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted if and only if every $(2k+2)$-block of $\alpha$ is $(k,p)$-sorted. The value $(2k+2)$ is optimal for the local characterization of the $(k,p)$-sortedness.
Proof. If $\alpha$ is $(k,p)$-sorted, then any block of $\alpha$ is $(k,p)$-sorted.

Conversely, let us assume that every $(2k+2)$-block of $\alpha$ is $(k,p)$-sorted. Let us define an order relation $\leq_p$ on any $(2k+2)$-block as follows. $a_j \leq_p a_i$ if and only if $a_j + p \leq a_i$ whenever $j + k < i$. Then this relation $\leq_p$ is transitive, and therefore $\alpha$ is $(k,p)$-sorted.

It can be verified by the following sequence that $(2k+2)$ is optimal. Every $(2k+1)$-block of the sequence is $(k,p)$-sorted, but $\alpha$ is not $(k,p)$-sorted. □

$$M(\alpha) = \begin{array}{c|cccc|} & 1 & 2 & p+1 & p+2 \\ \hline 1 & & 1 & & \\ 2 & 1 & & & \\ \vdots & \vdots & & & \\ k+1 & 1 & & & \\ k+2 & & & & 1 \\ \vdots & & & & \vdots \\ 2k+1 & & & & 1 \\ 2k+2 & & & 1 & \\ \hline \end{array}$$

Fig. 2.1

We can also give an example of $\alpha$ with all distinct elements that shows the optimal size to be $(2k+2)$ as follows.

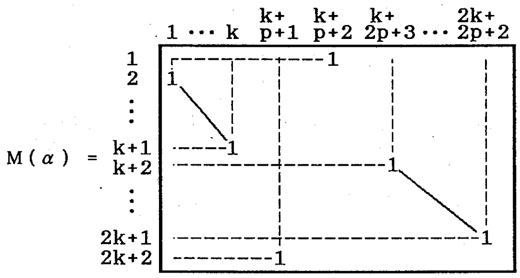


Fig. 2.2

When $\alpha = (a_1, a_2, \cdots, a_n)$ is a permutation of $n$ consecutive integers, the situation differs. Since any $n$ consecutive integers can be adopted, we assume that $\alpha$ is a permutation of $1, 2, \cdots, n$. In the rest of Section 2, we assume that $\alpha = (a_1, a_2, \cdots, a_n)$ is a permutation of $1, 2, \cdots, n$ unless stated otherwise.

**Lemma 2.2.** If $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, then $p \le k+1$, where $0 \le k \le n-2$.
Proof. Contrary assume that $p \ge k+2$. By the assumption, if $1+k < i$, then $a_1 + p \le a_i$. Since $a_1 \ge 1$, $a_i \ge p+1$. There are $(n-k-1)$ positions for such $a_i$, but there are $n-p(\le n-k-2)$ values to be selected. This is a contradiction and we have shown that $p \le k+1$. □

When $\alpha = (a_1, a_2, \cdots, a_n)$ is a permutation of $1, 2, \cdots, n$, $(2k+2)$ is also the optimal size of the blocks for the local characterization of $(k,p)$-sortedness for $p = 0, 1$ and $2$.

The optimality of $(2k+2)$ for $p = 0, 1$ and $2$ is shown by the similar example as in Fig 1.4.
For $p \ge 3$, the optimal size of the blocks for the local characterization of the $(k,p)$-sortedness of a sequence may be reduced as in the following results.

**Lemma 2.3.** Let $3 \le p \le k+1$ and $n \ge 2k+2$. If every $(2k+4-p+t)$-block of $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, then every $(2k+5-p+t)$-block of $\alpha$ is $(k,p)$-sorted, where $0 \le t \le p-3$.
Proof. From the assumption,

$$\left.\begin{array}{lllll}
a_1 + p & \le & a_{k+2}, & \cdots & , a_{2k+4-p+t} \\
a_2 + p & \le & a_{k+3}, & \cdots & , a_{2k+5-p+t} \\
\cdots & & & \cdots & \cdots \\
a_{n-2k-4+p-t} + p & \le & a_{n-k-3+p-t}, & \cdots, & a_{n-1} \\
a_{n-2k-3+p-t} + p & \le & a_{n-k-2+p-t}, & \cdots, & a_n \\
\cdots & & & \cdots & \cdots \\
a_{n-k-1} + p & \le & & & a_n
\end{array}\right\} \quad (1)$$

In order to prove the lemma, it is sufficient to show that

$$\left.\begin{array}{lll}
a_1 + p & \le & a_{2k+5-p+t} \\
a_2 + p & \le & a_{2k+6-p+t} \\
\cdots & & \cdots \\
a_{n-2k-4+p-t} + p & \le & a_n.
\end{array}\right\} \quad (2)$$

Let us contrary assume that at least one inequality in (2) does not hold. Let the last inequality in (2) which does not hold be the $x$-th one . Then we have

$$a_{x+2k+4-p+t} < a_x + p,$$
$$1 \le x \le n-2k-4+p-t. \qquad (3)$$

From (1), $a_{2k+4-p+t} \ge a_1+p, a_2+p, \cdots, a_{k+3-p+t}+p$. Therefore we have

$$a_{2k+4-p+t} \ge k+3+t \qquad (4)$$

Similarly we have

$$a_{2k+5-p+t}, \cdots, a_n \ge k+3+t \qquad (5)$$

Since $a_{3k+5-p+t} \ge a_{2k+4-p+t} + p$, we have

$$a_{3k+5-p+t} \ge k+3+t+p. \qquad (6)$$

Similarly,

$$a_{3k+6-p+t}, \cdots, a_n \ge k+3+t+p \qquad (7)$$

In general,

$$a_{2k+3-p+t+s(k+1)+1}, \cdots, a_{2k+3-p+t+(s+1)(k+1)}$$
$$\ge k+3+t+ps, \qquad (8)$$

where

$$s = 0, 1, \cdots, \lfloor \tfrac{n-2k-3+p-t}{k+1} \rfloor - 1 = N.$$

When $(k+1)$ does not divide $(n-2k-3+p-t)$,

$$a_{2k+3-p+t+(N+1)(k+1)+1}, \cdots, a_n$$
$$\ge k+3+t+p(N+1). \qquad (9)$$

Let us assume that

$$2k+3-p+t+s_0(k+1)+1 \le x+2k+4-p+t$$
$$\le 2k+3-p+t+(s_0+1)(k+1).$$

Then

$$a_{x+2k+4-p+t} \ge k+3+t+ps_0. \qquad (10)$$

From (1) and (3),

$$a_{x+2k+4-p+t} < a_{x+k+1}, \cdots, a_{x+2k+3-p+t}.$$

There are at least $(k+3-p+t)$ terms greater than $a_{x+2k+4-p+t}$, and we have

$$a_{x+2k+4-p+t} \le n-k-3+p-t. \qquad (11)$$

From (10) and (11),

$$k+3+t+ps_0 \le a_{x+2k+4-p+t}$$
$$\le n-k-3+p-t. \qquad (12)$$

Now, let

$$a_{x+2k+4-p+t} = k+3+t+ps_0+z, \qquad (13)$$
$$0 \le z \le n-2(k+3+t)+p-ps_0.$$

Since $a_{x+1}, \cdots, a_{x+k+3-p+t}$ are less than or equal to $a_{x+2k+4-p+t} - p$, at most $(ps_0+z)$ terms in $\{a_1, a_2, \cdots, a_{x-1}, a_{x+k+4-p+t}, \cdots, a_{n-k-1}\}$ can be less than or equal to $\{k+3+t+p(s_0-1)+z\}$. Then in the right hand side of inequalities in (1), at least $(n-k-2-ps_0-z)$ terms in $\{a_{k+2}, \cdots, a_{x+2k+3-p+t}, a_{x+2k+5-p+t}, \cdots, a_n\}$ are greater than $\{k+3+t+ps_0+z\}$. While there are $n-\{k+3+t+ps_0+z\}$ integers greater than $\{k+3+t+ps_0+z\}$. Since $\{n-(k+3+t+ps_0+z)\} < \{n-(k+2+ps_0+z)\}$, this contradicts the assumption that $a_1, \cdots, a_n$ are distinct. Hence, every $(2k+5-p+t)$-block of $\alpha$ is

$(k,p)$-sorted. □

By Lemma 2.3, we can prove the following theorem including the case for $p = 2$.

**Theorem 2.4.** Let $2 \leq p \leq k+1$ and $n \geq 2k+2$. $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted if and only if every $(2k+4-p)$-block is $(k,p)$-sorted. The value $(2k+4-p)$ is optimal for the local characterization of the $(k,p)$-sortedness.
Proof. The first half is a direct consequence of Lemma 2.3. The second half is shown by the following example, where every $(2k+3-p)$-block is $(k,p)$-sorted, but every $(2k+4-p)$-block is not $(k,p)$-sorted. □

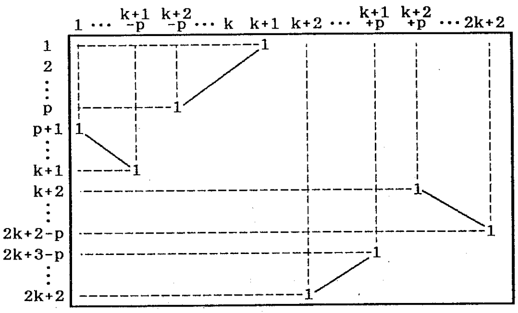


Fig. 2.3

## 2.2. A hierarchy of local characterizations of (k,p)-sorted sequences

In the previous section, we have investigated the problem of the optimal value of the block length for the local characterization. In this section, we will give another local characterization.

**Definition 2.5.** Let $\alpha = (a_1, a_2, \cdots, a_n)$ be a sequence of integers and $\Gamma = \{B_1, B_2, \cdots, B_m\}$ a set of blocks of $\alpha$. $\Gamma$ is called a *(k,p)-covering* of $\alpha$ if the $(k,p)$-sortedness of each element of $\Gamma$ implies the $(k,p)$-sortedness of $\alpha$.

**Lemma 2.6.** Let $2k+2 \leq n$ and $2 \leq p \leq k+1$. A set of blocks $\Gamma = \{B_1, B_2, \cdots, B_m\}$ of $\alpha = (a_1, a_2, \cdots, a_n)$ is a $(k,p)$-covering of $\alpha$ if and only if every $(2k+4-p)$-block of $\alpha$ is included in some element of $\Gamma$.
Proof. If every $(2k+4-p)$-block of $\alpha$ is included in some element of $\Gamma$, then $(k,p)$-sortedness of each element of $\Gamma$ means $(k,p)$-sortedness of each $(2k+4-p)$-block of $\alpha$ and by Theorem 2.4 $\alpha$ is $(k,p)$-sorted.
To prove only if part, let us assume that there is a $(2k+4-p)$-block of $\alpha$ which is not included in any element of $\Gamma$. Then $(k,p)$-sortedness of each element of $\Gamma$ dose not mean $(k,p)$-sortedness of each $(2k+4-p)$-block of $\alpha$. Therefore, $\alpha$ may not be $(k,p)$-sorted and $\Gamma$ may not be a $(k,p)$-covering of $\alpha$. □

Now we will consider a problem of the periodic location of blocks of a covering of a sequence.

**Definition 2.7.** A set of blocks $\Gamma = \{B_1, B_2, \cdots, B_m\}$ is said to be of *period* $\beta$ if $B_i$ is a block located at $1+(i-1)\beta$. A set of blocks $\Gamma = \{B_1, B_2, \cdots, B_m\}$ of period $\beta$ is said to be of *length* $\gamma$ if every block possibly except for the last one is of length $\gamma$ and when the last one is excepted, it has length smaller than $\gamma$.

**Lemma 2.8.** Let $2 \leq p \leq k+1$ and $2k+2 \leq n$. A set of blocks $\Gamma = \{B_1, B_2, \cdots, B_m\}$ of $\alpha = (a_1, a_2, \cdots, a_n)$ of period $\beta$ and length $\gamma$ is a $(k,p)$-covering of $\alpha$ if and only if every $a_i$ is contained in at least one element of $\Gamma$ and $|B_i \cap B_{i+1}| \geq (2k+3-p)$ for each $1 \leq i \leq m-1$, where $\beta \geq 1$ and $\gamma \geq 2$.
Proof. If $\Gamma = \{B_1, B_2, \cdots, B_m\}$ is a $(k,p)$-covering of $\alpha$, then every $(2k+4-p)$-block of $\alpha$ is included in some block. Therefore $|B_i \cap B_{i+1}| \geq (2k+3-p)$. Conversely, if $|B_i \cap B_{i+1}| \geq (2k+3-p)$, then every $(2k+4-p)$-block is included in some element in $\Gamma$ and $\Gamma$ is a $(k,p)$-covering of $\alpha$. □

**Theorem 2.9.** Let $2 \leq p \leq k+1$ and $2k+2 \leq n$. $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted if and only if each block of a set of blocks $\Gamma = \{B_1, B_2, \cdots, B_m\}$ of period $\beta$ and length $(2k+3-p+\beta)$ is $(k,p)$-sorted, where $\beta \geq 1$. The value $(2k+3-p+\beta)$ is optimal for the period $\beta$.
Proof. If $\alpha$ is $(k,p)$-sorted, then every $(2k+3-p+\beta)$-block of $\alpha$ is $(k,p)$-sorted. Hence, $B_1, B_2, \cdots, B_m$ are $(k,p)$-sorted. If every block of $\Gamma$ is $(k,p)$-sorted, then every $(2k+4-p)$-block of $\alpha$ is $(k,p)$-sorted, since $|B_i \cap B_{i+1}| \geq (2k+3-p)$ for $1 \leq i \leq m-1$. Hence, $\alpha$ is $(k,p)$-sorted. If the length is smaller than $(2k+3-p+\beta)$, then there exists a $(2k+4-p)$-block of $\alpha$ which is not included in any element in $\Gamma$. Therefore, the value $(2k+3-p+\beta)$ is optimal. □

## 2.3. Recursive characterization

When $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, we define a subsequence $\alpha_i$ of $\alpha$ as follows;

$\alpha_i = (a_i, a_{i+k+1}, a_{i+2(k+1)}, \cdots)$,

where $1 \leq i \leq k+1$.
Then $\alpha_i$ is a completely sorted sequence.
Represent each $\alpha_i$ $(1 \leq i \leq k+1)$ on a straight line segment and join $a_s$ and $a_t$ when $s+k<t$ and delete redundant lines. The resulting reduced representation is called the *periodic representation* of $\alpha$ and denoted by $N^n_{k,p}(\alpha)$. An example of $N^n_{k,p}(\alpha)$ is shown as follows.

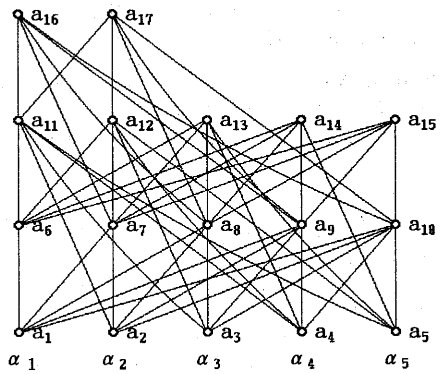


Fig. 2.4 $N^{17}_{4,p}(\alpha)$

A subset $N^n_{k,p}(\alpha;\bar{i})$ of $N^n_{k,p}(\alpha)$, $1 \le k$ and $1 \le i \le k+1$, is defined as follows.

$N^n_{k,p}(\alpha;\bar{i}) = N^n_{k,p}(\alpha) - \alpha_i$.

The subsequence of $\alpha$ defined by $N^n_{k,p}(\alpha;\bar{i})$ is denoted by $\alpha(\bar{i})$.

**Proposition 2.10.** Let $n \ge 2k+2$. If $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted $(k \ge 1)$, then $\alpha(\bar{i})$ is $(k-1,p)$-sorted for any $i, 1 \le i \le k+1$.
Proof. Omitted.

From this property, we can obtain two kinds of recursive characterizations for the $(k,p)$-sortedness.

**Theorem 2.11.** Let $n \ge 2k+2$. $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted $(k \ge 2)$ if and only if for any $i, 1 \le i \le k+1$, every $\alpha(\bar{i})$ is $(k-1,p)$-sorted.
Proof. Omitted.

**Theorem 2.12.** Let $n \ge 2k+2$. $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted $(k \ge 1)$ if and only if $\alpha(\bar{1})$ and $\alpha(\overline{k+1})$ are $(k-1,p)$-sorted and the following conditions are satisfied;

(i) for any element $a_s$ in $\alpha_1$, the element $a_{s+2k+1}$ in $\alpha_{k+1}$ satisfies $a_s + p \le a_{s+2k+1}$,
(ii) for any element $a_t$ in $\alpha_{k+1}$, the element $a_{t+k+2}$ in $\alpha_1$ satisfies $a_t + p \le a_{t+k+2}$.
Proof. Omitted.

Now, we define a subset $N^n_{k,p}(\alpha;i,j)$ of $N^n_{k,p}(\alpha)$ as a periodic subrepresentation of $N^n_{k,p}(\alpha)$ induced by $\alpha_i, \alpha_{i+1}, \cdots, \alpha_j$, where $1 \le i \le j \le k+1$. When $i = j$, $N^n_{k,p}(\alpha;i,j)$ is denoted by $N^n_{k,p}(\alpha;i)$. From Theorem 2.12, the recursive structure of $N^n_{k,p}(\alpha)$ is shown in Fig. 2.5.

## 3. Classification of (k,p)-sorted sequences

Through this section we assume that $\alpha = (a_1, a_2, \cdots, a_n)$ is a permutation of $1, 2, \cdots, n$.

**Proposition 3.1.** Let $0 \le k \le \lfloor \frac{1}{2}n \rfloor - 1$. If a sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k, k+1)$-sorted, then $\alpha$ is completely sorted.
Proof. Contrary assume that $\alpha$ is not completely sorted. Then there exists an integer $j$ such that $a_j \ne j$. Let j be the smallest one satisfying the condition. Then $a_j > j$ and $a_i = i$ for all $1 \le i < j$. If $1 \le j \le \frac{1}{2}n$, then $a_s \ge a_j + k + 1$ for all $s > j+k$. Number of such elements is $(n - a_j - k)$. From the definition of $(k,k+1)$-sortedness, $(n - a_j - k)$ is at least $(n - j - k)$. Therefore, $n - a_j - k \ge n - j - k$. Hence, we have $a_j \le j$. This is a contradiction. If $j > \frac{1}{2}n$, then there is an integer $t > j$ such that $a_t < t$. Let $t$ be the smallest one satisfying the condition. If $t > j+k$, then $a_t$ satisfies $a_t \ge a_{t-k-1} + (k+1)$ $\ge (t-k-1) + (k+1) = t$. This is a contradiction. If $j < t \le j+k$, then $a_t$ satisfies $a_t \ge a_{t-k-1} + (k+1)$ $= (t-k-1)+(k+1) = t$. This is also a contradiction. Hence, $\alpha$ is completely sorted. $\square$

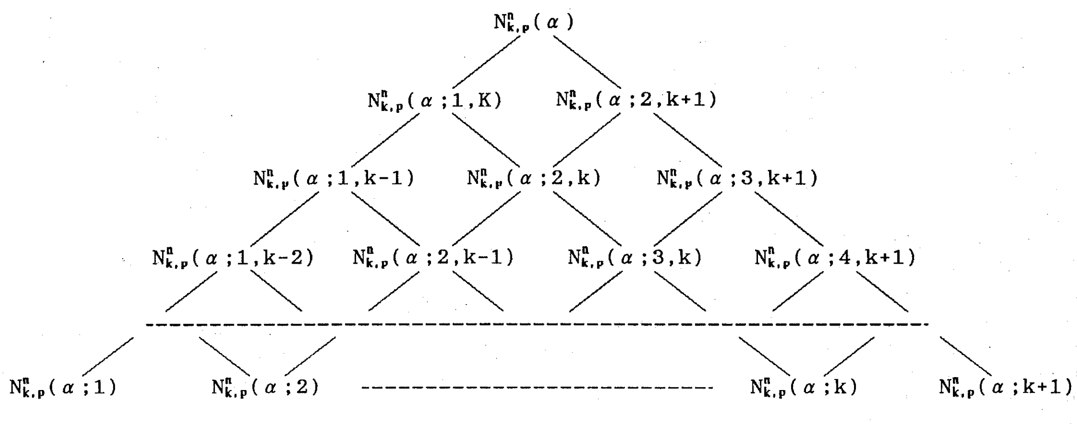


Fig. 2.5 Recursive structure of $N^n_{k,p}(\alpha)$

For $k \geq \lfloor \frac{1}{2}n \rfloor$, we need more definitions.

**Definition 3.2.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(c_1, c_2)$*-periphery sorted*, if for each $i$ $(1 \leq i \leq c_1)$ and $j$ $(n - c_2 + 1 \leq j \leq n)$ we have $a_i = i$ and $a_j = j$, where $c_1 + c_2 \leq n$. When $c_1 = c_2 = c$, $\alpha$ is said to be $c$*-periphery sorted.*

**Proposition 3.3.** If a sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k, k+1)$-sorted, where $\lfloor \frac{1}{2}n \rfloor \leq k \leq n-1$, then $\alpha$ is $(n-k-1)$-periphery sorted.
Proof. If an element $a_i$ has an element to be compared, then $i+k+1 \leq n$ or $i-k-1 \geq 1$. Therefore $1 \leq i \leq n-k-1$ or $k+2 \leq i \leq n$.
If $1 \leq i \leq n-k-1$, then for any $j > i+k$, we have $a_j \geq a_i+k+1$. The number of elements greater than or equal to $a_i+k+1$ is $(n-a_i-k)$ and this is at least $n-i-k$. Then we have $n-a_i-k \geq n-i-k$.
Hence, $a_i \leq i$ for all $i$, $1 \leq i \leq n-k-1$.
Therefore, $a_i = i$ for all $i$, $1 \leq i \leq n-k-1$.
Similarly, $a_j = j$ for all $j$, $k+2 \leq j \leq n$.
Hence, $\alpha$ is $(n-k-1)$-periphery sorted. □

**Definition 3.4.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is said to be $(s_1, s_2;d)$*-semideviation sorted* if for $s_1 < i$, $i - a_i \leq d$ and for $j < n - s_2 + 1$, $a_i - i \leq d$. When $s_1 = s_2 = s$, $\alpha$ is said to be $(s;d)$*-semideviation sorted.*

**Proposition 3.5.** Let $0 \leq k \leq \lfloor \frac{1}{2}n \rfloor - 1$ and $1 \leq p < k+1$. If a sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, then $\alpha$ is $(k+1;k+1-p)$-semideviation sorted.
Proof. For any $i$, $k+1 < i \leq n$, term $a_i$ has a term to be compared on the left. For any $t$, $t+k < i$, $a_t + p \leq a_i$. The number of integers less than or equal to $a_i - p$ is $a_i - p$, and this is not smaller than $i-k-1$. Then we have $a_i - p \geq i-k-1$.
Hence, $i - a_i \leq k+1-p$.
Similarly, for $j < n-k$, we have $a_j - j \leq k+1-p$.
Therefore, $\alpha$ is $(k+1, k+1-p)$-semideviation sorted. □

**Definition 3.6.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is said to be $d$*-semideviation* $(c_1, c_2)$*-periphery sorted* if for $1 \leq i \leq c_1$, $a_i - i \leq d$ and for $n - c_2 + 1 \leq j \leq n$, $j - a_j \leq d$.

**Proposition 3.7.** Let $\lfloor \frac{1}{2}n \rfloor \leq k \leq n-1$ and $1 \leq p < k+1$. If a sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, then $\alpha$ is $(k+1-p)$-semideviation $(n-k-1)$-periphery sorted.
Proof. For any $i$, $1 \leq i \leq n-k-1$, the term $a_i$ has an element to be compared on the right. For any $t$ such that $i+k < t$, we have $a_i+p \leq a_t$. The number of terms larger than or equal to $a_i+p$ is $(n-a_i-p+1)$ and this is at least $(n-k-i)$.
Hence, $a_i - i \leq k+1-p$.
Similarly, for any $j$, $k+2 \leq j \leq n$, we have $j - a_j \leq k+1-p$.
Therefore $\alpha$ is $(k+1-p)$-semideviation $(n-k-1)$-periphery sorted. □

**Definition 3.8.** A sequence $\alpha = (a_1, a_2, \cdots, a_n)$ is $d$*-deviation sorted* if $| a_i - i | \leq d$ for any $i$.
A sequence $\alpha$ is $d$*-deviation,* $c$*-periphery sorted* if for $i$, $1 \leq i \leq c$ and $j$, $n-c+1 \leq j \leq n$ we have $| a_i - i | \leq d$ and $| a_j - j | \leq d$.

**Proposition 3.9.** For $p = 0$ or $1$, the following properties are valid;
(i) for $0 \leq k \leq \lfloor \frac{1}{2}n \rfloor - 1$, if $\alpha = (a_1, a_2, \cdots, a_n)$ is $(k,p)$-sorted, then $\alpha$ is $k$-deviation sorted,
(ii) for $\lfloor \frac{1}{2}n \rfloor \leq k \leq n-1$, if $\alpha$ is $(k,p)$-sorted, then $\alpha$ is $k$-deviation, $(n-k-1)$-periphery sorted.
Proof. (i) Since $\alpha = (a_1, a_2, \cdots, a_n)$ is a permutation of $(1, \cdots, n)$, $a_i \neq a_j$ whenever $i \neq j$. Therefore, $a_j \leq a_i$ whenever $j+k < i$ is equivalent to $a_j+1 \leq a_i$ whenever $j+k < i$. Hence, we assume that $p = 1$.
For any $i$, $k+1 < i \leq n$, the term $a_i$ has a term to be compared on the left. For any $t$ such that $t+k < i$, $a_t + 1 \leq a_i$. Number of integers less than or equal to $a_i - 1$ is $a_i - 1$ and this is not smaller than $i-k-1$. Then we have $a_i - 1 \geq i-k-1$.
Hence, $i - a_i \leq k$.
For any $i$, $1 \leq i \leq k+1$, $i - a_i \leq i - 1 \leq k$ since $a_i \geq 1$.
Therefore, for any $i$, $1 \leq i \leq n$, $i - a_i \leq k$.
Similarly we can show that $a_i - i \leq k$ for any $i$, $1 \leq i \leq n$. This implies that $\alpha$ is $k$-deviation sorted.
(ii) Similarly proved as in (i). □

Results in section 3 are resumed in Fig. 3.1.

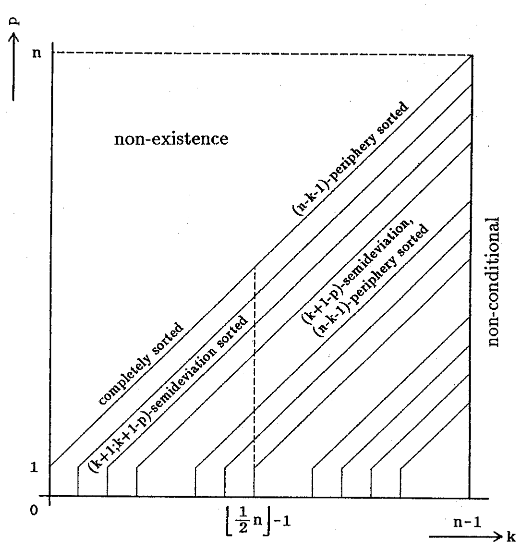


Fig. 3.1

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