ハイパーリングのハイパーキューブへの埋め込み

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グラフ $G=(V,E)$ が，$V=\{0,\cdots,N-1\}, E=\{(u,v)\mid(u-v)$ modulo Nが 2 の冪乗$\}$ であるとき，このグラフを頂点数 N のハイパーリング (N-HR) という．本稿では，2^n-HR の n 次元ハイパーキューブへの埋め込みについて論じる．はじめに，ディレイション，コンジェスチョンがそれぞれ 2，$O(n)$ である貪欲埋め込みを示す．次に，この貪欲埋め込みをモディファイし，ディレイション 4，コンジェスチョン 7 の埋め込みを得ることを示す．

Embedding of the Hyper-Ring into the Hypercube

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A graph $G=(V,E)$ with N nodes is called an N-hyper-ring if $V=\{0,\cdots,N-1\}$ and $E=\{(u,v)\mid(u-v)$ modulo N is a power of $2\}$. We study embedding of the 2^n-hyper-ring into the n-dimensional hypercube. Firstly, we show a greedy embedding whose dilation and congestion are 2 and $O(n)$, respectively. Then we modify the greedy embedding to achieve an embedding with dilation 4 and congestion 7.

1 Introduction

Embedding of one graph into another has been studied extensively[1–7]. Embedding is an important subject for many problems in computer science such as simulating one parallel processing architecture by another, assigning processors in a distributed system, and simulating one data structure by another. The efficiency of such a simulation can be measured by the load, expansion, dilation, and congestion of an embedding.

A hyper-ring was discussed in [2], where constructions, properties, containments, and spanners of hyper-rings are described. In particular, the 2^n-hyper-ring contains the n-dimensional hypercube, any dimensional 2^n-node mesh, and the $(2^n - 1)$-node complete binary tree as subgraphs.

A hypercube is one of the most popular networks due to its recursive structure, low diameter, high bisection width, and existence of simple algorithms for message routing on it. There exists, however, some problems which have not yet been solved on the hypercube[3,6]. In this report, we show the embedding of the 2^n-hyper-ring into the n-dimensional hypercube with small constant dilation and congestion. It turns out to show that the hypercube and the hyper-ring are computationally equivalent. If we can solve a problem on the hyper-ring, then it can be also solved on the hypercube by the same algorithm with a minor modification and a small constant factor of slowdown in computing time.

This report is organized as follows. We give some definitions in Section 2. In Section 3, we describe a greedy embedding of the 2^n-hyper-ring into the n-dimensional hypercube, whose dilation and congestion are 2 and $O(n)$, respectively. In Section 4, we modify the greedy embedding described in Section 3 to achieve an embedding with dilation 4 and congestion 7.

2 Preliminaries and Definitions

A graph $G = (V, E)$ with N nodes is called an *N-hyper-ring* (N-HR for short) if $V = \{0, 1, \cdots, N-1\}$ and $E = \{(u, v) \mid (u - v) \text{ modulo } N \text{ is a power of } 2\}$. The *$n$-dimensional hypercube* (2^n-HC for short) is a graph with 2^n nodes labeled by all n-bit binary numbers, where there exists an edge between two nodes if and only if their binary representations differ in an exactly one bit.

Let $G = (V, E)$ and $H = (W, F)$ be graphs. Let P be the set of paths between any two nodes in H. An embedding of G(a guest-graph) into H(a host-graph) is a pair of two mappings $\sigma : V \to W$ and $\rho : E \to P$ such that for all egdes (u, v) in E, $\rho((u, v))$ connects $\sigma(u)$ and $\sigma(v)$. Efficiency of an embedding is measured by its *load*, *expansion*, *dilation*, and *congestion*. The load of an embedding is the largest number of nodes mapped to a single node. The expansion is the ratio of the cardinarity of the node set of the host-graph to the one of the guest-graph. The dilation of an edge e in G under an embedding is the length of the path $\rho(e)$. The dilation of an embedding is the maximum dilation of any edge in G. The congestion of an edge f in H under an embedding is the number of edges e in G such that $\rho(e)$ contains f. The congestion of an embedding is the maximum congestion of any edge in H.

Next we define a Gray code which is used to correspond nodes between the 2^n-HR and the 2^n-HC. An n-bit *Gray code* is an ordering of all n-bit binary numbers so that a pair of consecutive numbers differ in exactly one bit position. We denote the ith codeword of the Gray code by $x(i)$. Formally, $x(i)$ is defined as follows.

$$x(i) = \begin{cases} 0 & \text{if } i = 0 \\ 2^t \oplus x(2^{t+1} - 1 - i) & \text{if } i > 0, \end{cases}$$

where $t = \lfloor \log i \rfloor$ and $\oplus$ denotes the *bitwise exclusive-or* operation. We prove that a pair of consecutive numbers differ in exactly one bit position in Lemma 1. We denote the least significant bit of the binary representation of i by $b_0(i)$, and the second least significant bit by $b_1(i)$, and so on. Throughout this report, n denotes a positive integer. For integers, a and b, $[a]_b$ denotes a *modulo* b. For a nonnegative integer i, we define

$$r_k(i) = \begin{cases} n-1 & \text{if } k \geq n \\ k & \text{if } k < n \text{ and } b_k(i) = 0 \\ r_{k+1}(i) & \text{otherwise.} \end{cases}$$

3 Greedy Embedding

In this section, we describe a greedy embedding which is determined by a bijection σ from the node set of the 2^n-HR to the one of the 2^n-HC, together with an injection ρ from the edge set of the 2^n-HR to the set of shortest paths that connect any two nodes in the 2^n-HC. We select x, which defines a Gray code, as σ and let ρ map any edge (i, j) in the 2^n-HR to one of the shortest paths (if more than one exists) that connect $x(i)$ and $x(j)$ in the 2^n-HC. Firstly, we describe properties of the n-bit Gray code to show that the dilation of the greedy embedding is 2.

Lemma 1 *Let i be an arbitrary integer such that $0 \leq i < 2^n$. Then, $x(i) \oplus x([i+1]_{2^n}) = 2^{r_0(i)}$.*

Proof: This lemma is proved by induction on t with $0 \leq i < 2^t$.

(Base case) For $t = 0$ $(0 \leq i < 2^0)$, we have $x(i) \oplus x([i+1]_{2^n}) = x(0) \oplus x(1) = 2^0$ and $r_0(0) = 0$ by the definitions of $x(i)$ and $r_k(i)$.

(Induction step) Let t be a nonnegative integer with $t < n$. Assume that the assertion holds for all i with $0 \leq i < 2^t$. We consider the following 3 cases: $2^t \leq i < 2^{t+1} - 1$, $i = 2^{t+1} - 1$ $(t < n-1)$, and $i = 2^n - 1$.

case 1 ($2^t \le i < 2^{t+1} - 1$): From the definition of x and the induction hypothesis the following equality is derived.

$$\begin{aligned} x(i) \oplus x([i+1]_{2^n}) &= (2^t \oplus x(2^{t+1} - 1 - i)) \oplus (2^t \oplus x(2^{t+1} - 1 - (i+1))) \\ &= x(2^{t+1} - 1 - i) \oplus x(2^{t+1} - 1 - (i+1)) \\ &= 2^{r_0(2^{t+1}-1-(i+1))}. \end{aligned}$$

We show $r_0(2^{t+1} - 1 - (i+1)) = r_0(i)$. Since $b_j(i) = 1$ for all j with $0 \le j < r_0(i)$, we have $b_j(i+1) \ne b_j(i)$ for all j with $0 \le j \le r_0(i)$. It follows that $b_j(2^{t+1} - 1 - (i+1)) = b_j(i)$ for all j with $0 \le j \le r_0(i)$ because $b_j(2^{t+1} - 1 - (i+1)) \ne b_j(i+1)$ for all j with $0 \le j \le r_0(i)$. Thus we have $r_0(2^{t+1} - 1 - (i+1)) = r_0(i)$.

case 2 ($i = 2^{t+1} - 1$, $t < n-1$): From the definition of x, we have $x(i) \oplus x([i+1]_{2^n}) = x(2^{t+1} - 1) \oplus x(2^{t+1}) = x(2^{t+1} - 1) \oplus (2^{t+1} \oplus x(2^{t+1} - 1)) = 2^{t+1}$. Since $r_0(2^{t+1} - 1) = t+1$, the assertion holds.

case 3 ($i = 2^n - 1$): From the definitions of x and $r_k(i)$, we have $x(i) \oplus x([i+1]_{2^n}) = x(2^n - 1) \oplus x(0) = 2^{n-1}$ and $r_0(2^n - 1) = n - 1$. □

Lemma 2 *Let i and k be arbitrary integers such that $0 \le i < 2^n$ and $1 \le k < n$, respectively. Then, $x([i + 2^k]_{2^n}) = x(i) \oplus 2^{k-1} \oplus 2^{r_k(i)}$.*

Proof: We prove this lemma by induction on k ($k \ge 1$).

(Base case) We show $x([i+2]_{2^n}) = x(i) \oplus 2^0 \oplus 2^{r_1(i)}$. From Lemma 1, we have

$$\begin{aligned} x([i+2]_{2^n}) &= x([[i+1]_{2^n} + 1]_{2^n}) \\ &= x(i) \oplus 2^{r_0(i)} \oplus 2^{r_0([i+1]_{2^n})}. \end{aligned}$$

If $b_0(i) = 0$ ($b_0([i+1]_{2^n}) = 1$), then $r_0(i) = 0$ and $r_0([i+1]_{2^n}) = r_1(i)$ from the definition of $r_k(i)$. Otherwise, $r_0(i) = r_1(i)$ and $r_0([i+1]_{2^n}) = 0$.

(Induction step) Assume that the assertion holds for $k < n-1$. Then, we have

$$\begin{aligned} x([i+2^{k+1}]_{2^n}) &= x([[i+2^k]_{2^n} + 2^k]_{2^n}) \\ &= x(i) \oplus 2^{r_k(i)} \oplus 2^{r_k([i+2^k]_{2^n})}. \end{aligned}$$

If $b_k(i) = 0$ ($b_0([i+2^k]_{2^n}) = 1$), then $r_k(i) = k$ and $r_k([i+2^k]_{2^n}) = r_{k+1}(i)$, otherwise $r_k(i) = r_{k+1}(i)$ and $r_k([i+2^k]_{2^n}) = k$. □

If we let ρ map any edge (i, j) in the 2^n-HR to one of the shortest paths (if more than one exists) that connect $x(i)$ and $x(j)$ in the 2^n-HC, the dilation of the greedy embedding is 2. And we have known that for any node i in the 2^n-HR and $k \ge 1$, $x(i)$ and $x([i+2^k]_{2^n})$ are bitwise equal except for the $(k-1)$-st bit and the $r_k(i)$-th bit. So there are two shortest paths that connect $x(i)$ and $x([i+2^k]_{2^n})$ for $k \geq 1$ in the 2^n-HC, which are the path from $x(i)$ to $x([i+2^k]_{2^n})$ using the $(k-1)$-dimensional edge first and then the $r_k(i)$-dimensional edge and the path using these two dimensions in reverse order. Let us call the former path a $(k-1)$-path and the latter one an $r_k(i)$-path. If we let ρ map any edge in the 2^n-HR to the corresponding $(k-1)$-path, we can show that the congestion becomes $O(n)$ using Lemma 3 described in the next section. It is trivial that the congestion becomes $O(n)$ if we let ρ map any edge in the 2^n-HR to the corresponding $r_k(i)$-path. It seems hard to determine the mapping ρ such that it maps each edge in the 2^n-HR to the either shortest path according to the value of i and k and that the embedding with constant congestion can be achieved.

From Lemma 1, Lemma 2, and the description above, we have the following theorem.

Theorem 1 *The dilation and the congestion of the greedy embedding of the 2^n-HR into the 2^n-HC are 2 and $O(n)$, respectively.* □

4 Embedding with Constant Dilation and Congestion

In this section, we modify the embedding described in the previous section and show that the dilation and the congestion of the modified one are 4 and 7, respectively. Here, we adopt x, which defines a Gray code, as σ again. To describe a mapping ρ in brief, we introduce the following notation. In the 2^n-HC, $\langle v; d_1, d_2, \cdots, d_m\rangle$ denotes the path from a node v to the node which is exactly different from v in the d_1-th, the d_2-th, $\cdots$, the d_m-th bits, using the d_1-dimensional edge first and then using the d_2-dimensional edge from the current node, and so on.

We define ρ as follows. Let $e = (i, [i+2^k]_{2^n})$ be an arbitrary edge in the 2^n-HR.

$$\rho(e) = \begin{cases} \langle x(i); r_0(i)\rangle & \text{if } k = 0 \\ \langle x(i); k-1, k\rangle & \text{if } k \neq 0, r_k(i) = k \\ \langle x(i); k-1, k, r_k(i), k\rangle & \text{if } k \neq 0, r_k(i) > k. \end{cases}$$

Next lemma says intuitively that for any two adjacent nodes $x(i)$ and $(x(i) \oplus 2^k)$ in the 2^n-HC, $x^{-1}(x(i) \oplus 2^k) = i \oplus (2^{k+1}-1)$, where i denotes a node in the 2^n-HR.

Lemma 3 *Let i and k be arbitrary integers such that $0 \leq i < 2^n$ and $0 \leq k < n$. Then, $x(i) \oplus x(i \oplus (2^{k+1}-1)) = 2^k$.*

Proof: We prove this lemma by induction on k $(k \geq 0)$.

(Base case) If $b_0(i) = 0$, $i \oplus (2^1 - 1) = i \oplus 2^0 = i + 1$. From Lemma 1, we have

$$x(i) \oplus x(i \oplus (2^1-1)) = x(i) \oplus x(i+1) = 2^{r_0(i)} = 2^0.$$

Next, we consider the case $b_0(i) = 1$. Let $i' = i \oplus 2^0$. Then $b_0(i') = 0$. According to the argument above, we have $x(i') \oplus x(i' \oplus 2^0) = 2^0$. Since $i' = i \oplus 2^0$ and $i' \oplus 2^0 = i$, $x(i \oplus 2^0) \oplus x(i) = 2^0$.

(Induction step) Assume that the assertion holds for $k < n-1$. By the induction hypothesis, we have

$$\begin{aligned} x(i) \oplus x(i \oplus (2^{k+2}-1)) &= x(i) \oplus x(i \oplus 2^{k+1} \oplus (2^{k+1}-1)) \\ &= x(i) \oplus x(i \oplus 2^{k+1}) \oplus 2^k. \end{aligned}$$

If $b_{k+1}(i) = 0$, then $i \oplus 2^{k+1} = i + 2^{k+1}$. From Lemma 2, we have

$$\begin{aligned} x(i) \oplus x(i \oplus (2^{k+2}-1)) &= x(i) \oplus x(i + 2^{k+1}) \oplus 2^k \\ &= 2^k \oplus 2^{r_{k+1}(i)} \oplus 2^k = 2^{r_{k+1}(i)}. \end{aligned}$$

Since $b_{k+1}(i) = 0$, $r_{k+1}(i) = k+1$. In the case $b_{k+1}(i) = 1$, let $i' = i \oplus 2^{k+1}$. Then, in a similar way above, we derive $x(i) \oplus x(i \oplus (2^{k+2}-1)) = x(i) \oplus x(i \oplus 2^{k+1}) \oplus 2^k = x(i' \oplus 2^k) \oplus x(i') \oplus 2^k = 2^{r_{k+1}(i')} = 2^{k+1}$. □

Lemma 4 *Let e* $= (x(i), x(i) \oplus 2^d)$ *be an arbitrary edge in the* 2^n*-HC. Then, the congestion of e under the embedding* (x, ρ) *is at most* 7.

Proof: From the definition of ρ, there is a possibility of 7 types of paths containing e. We refer to each of them as *type 1*, $\cdots$, *type 7* according to Figure 1, where the bold edge stands for e. Firstly, we show that the number of edges in the 2^n-HR mapped to each

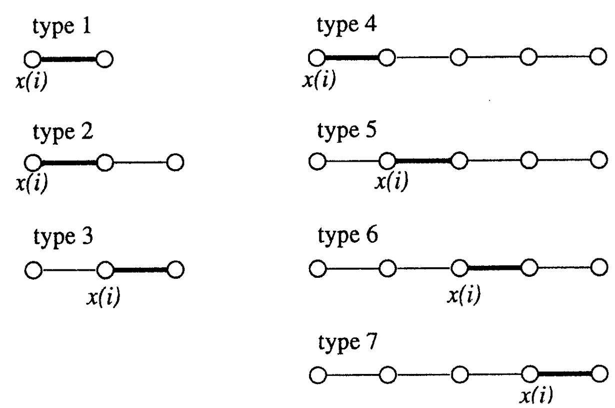

Figure 1: 7 types of paths

path of 7 types which emanates from $x(i)$ in the 2^n-HC is exactly 1. Note that each edge

$(i, [i+2^k]_{2^n})$ in the 2^n-HR is specified by its endpoints i and $[i+2^k]_{2^n}$, in turn by i and k. In analyzing each path of 7 types, we can characterize the value of $r_d(i)$ or $r_{d+1}(i)$.

type 1: If e is contained in the path p_1 of type 1, then $\rho^{-1}(p_1) = (i, [i+1]_{2^n})$. Note that $r_0(i) = d$.

type 2: If e is contained in the path p_2 of type 2, then $\rho^{-1}(p_2) = (i, [i+2^{d+1}]_{2^n})$. Note that $r_{d+1}(i) = d+1$.

type 3: If e is contained in the path p_3 of type 3, then by Lemma 3, $\rho^{-1}(p_3) = (i \oplus (2^d - 1), [(i \oplus (2^d-1)) + 2^d]_{2^n})$. Note that $r_d(i) = d$ since $r_d(i \oplus (2^d - 1)) = d$.

type 4: If e is contained in the path p_4 of type 4, then $\rho^{-1}(p_4) = (i, [i+2^{d+1}]_{2^n})$. Note that $r_{d+1}(i) > d+1$.

type 5: If e is contained in the path p_5 of type 5, then by Lemma 3, $\rho^{-1}(p_5) = (i \oplus (2^d - 1), [(i \oplus (2^d-1)) + 2^d]_{2^n})$. Note that $r_d(i) > d$ since $r_d(i \oplus (2^d - 1)) > d$.

type 6: Let e be contained in the path p_6 of type 6. In this case, by Lemma 3, it is obtained that for some $k > 0$, $\rho^{-1}(p_6) = (i \oplus 2^k, [(i \oplus 2^k) + 2^k]_{2^n})$ and $r_k(i \oplus 2^k) = d > k$. However, it is not trivial that there is exactly one edge which satisfies these conditions. To derive a contradiction, we assume that there are two edges $(i \oplus 2^k, [(i \oplus 2^k) + 2^k]_{2^n})$ and $(i \oplus 2^{k'}, [(i \oplus 2^{k'}) + 2^{k'}]_{2^n})$ such that $r_k(i \oplus 2^k) = d > k$ and $r_{k'}(i \oplus 2^{k'}) = d > k'$. Without loss of generality, we assume $k < k'$. Here $(i \oplus 2^k)$ and $(i \oplus 2^{k'})$ are bitwise equal except for the k-th bit and the k'-th bit. So we have $b_{k'}(i \oplus 2^k) \neq b_{k'}(i \oplus 2^{k'})$. However, $b_{k'}(i \oplus 2^k) = 1$ and $b_{k'}(i \oplus 2^{k'}) = 1$ since $r_k(i \oplus 2^k) = d > k' > k$ and $r_{k'}(i \oplus 2^{k'}) = d > k'$. This is a contradiction. Note that $r_d(i) = d$ since $r_k(i \oplus 2^k) = d > k$.

type 7: Let e be contained in the path p_7 of type 7. From the definition of ρ and Lemma 3, we know that for some u, $\rho^{-1}(p_7) = (u, [u+2^d]_{2^n})$, $u = i \oplus 2^d \oplus (2^{r_d(u)+1} - 1)$, and $r_d(u) > d$. If we assume that there exists another node $u' \neq u$ in the 2^n-HR such that $u' = i \oplus 2^d \oplus (2^{r_d(u')+1} - 1)$ and $r_d(u') > d$, we can derive a contradiction in a similar way described for type 6. Note that $r_d(i) > d$ since $r_d(i \oplus 2^d \oplus (2^{r_d(u)} - 1) > d$ and $r_d(u) > d$.

Now we count the number of paths which contain an edge $e = (x(i), x(i) \oplus d)$ in the 2^n-HC. Let $i' = i \oplus (2^{d+1} - 1)$, then $x(i)$ and $x(i')$ are the endpoints of e.

If $n = 1$, it is trivial that both the dilation and the congestion of the embedding described are 1.

Assume that $n > 1$. There is at most 1 path of type 1 containing e since $r_0(i) = d$ if and only if $r_0(i') \neq d$.

For the other types, we consider the following 2 cases: $d < n-1$ and $d = n-1$.

case 1 ($d < n-1$): In this case, note that $r_d(i) = d$ if and only if $r_d(i') > d$ and that $r_{d+1}(i) = r_{d+1}(i')$. Without loss of generality, we assume $r_d(i) = d$ ($r_d(i') > d$). Then, there are at most 2 paths emanating from $x(i)$, one of which is type 3 and the other is type 6. There are also at most 2 paths emanating from $x(i')$, one of which is type 5 and the other is type 7.

If $r_{d+1}(i) = r_{d+1}(i') = d+1$, there are at most 2 paths of type 2, one of which is emanating from $x(i)$ and the other is from $x(i')$, and there are no paths of type 4. Otherwise, there are at most 2 paths of type 4 in a similar way.

To sum up for $d < n-1$, at most 7 paths are embedded using e.

case 2 ($d = n-1$): In this case, note that $r_d(i) = r_d(i') = d$ and that we need not consider the paths of type 2, type 4, type 5, and type 7 from the conditions of $r_d(i), r_d(i')$ or $r_{d+1}(i), r_{d+1}(i')$. There are at most 2 paths of type 3, one of which is emanating from $x(i)$ and the other is from $x(i')$. There are also at most 2 paths of type 6 in a similar way.

To sum up for $d = n-1$, at most 5 paths are embedded using e. □

From the definition of ρ and Lemma 4, we have the following theorem.

Theorem 2 *The 2^n-HR is embedded into the 2^n-HC with dilation 4 and congestion 7.* □

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