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放物線のアレンジメントの組合せ複雑度について

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曲線族のアレンジメントの複雑度の評価は計算幾何学で重要な問題である。直線のアレンジメントに関しては多くの結果があるが、その他の曲線のアレンジメントに対しての研究は困難性が高く、あまり進んでいない。本論文では、疑似放物線族のアレンジメントの複雑度を考察する。

二次関数のグラフ、即ち平面上の y軸方向の放物線の族を考えよう。各々の放物線は平面を二分し、二本の放物線は互いに高々２回しか交わらない。一般に、それぞれが平面を二分し、また、互いに２回ずつしか交わらない平面曲線の族を疑似放物線族という。

主結果としては、疑似放物線族を切断して互いに１回しか交わらない曲線分の族にすることを考え、必要な最小切断数の下界 $O(n^{4/3})$ と上界 $O(n^{5/3})$ を与える。また、円のアレンジメントに対しても同じ下界と上界を与える。応用として、疑似放物線のアレンジメントのレベルの複雑度に対する $O(n^{23/12})$ の新しい上界を与え、又、移動する点のユークリッド最小木の組合せ変更数の既知の上界を $O(n^{1/12})$ 改良する。

How to cut pseudo-parabolas into segments

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An arrangement of curves is a major research target in computational geometry. Although many results are known on complexities of arrangement of lines, few nontrivial corresponding results are known for other classes of curves. In this paper, we consider combinatorial complexity of an arrangement of *pseudo-parabolas*.

Let Γ be a collection of unbounded Jordan arcs intersecting at most twice each other, which we call pseudo-parabolas, since two axis parallel parabolas intersects at most twice. We investigate how to cut pseudo-parabolas into the minimum number of curve segments so that each pair of segments intersect at most once. We give an $\Omega(n^{4/3})$ lower bound and $O(n^{5/3})$ upper bound. We give the same bounds for an arrangement of circles. Applying the upper bound, we give an $O(n^{23/12})$ bound on the complexity of a level of pseudo-parabolas, and $O(n^{3-1/12}\alpha(n))$ bound on the number of combinatorial transitions of L_2 Euclidean minimum spanning trees of linearly moving points.

1 Introduction

Arrangement of curves in a plane is a major research target in computational geometry. Combinatorial complexities of parts of arrangements such as a cell, many cells, k-levels, $\leq k$-levels, and peaks of k-levels play key roles in designing algorithms on geometric optimization and motion planning problems [4, 7, 12].

Although arrangements of lines and line segments are most popular, an arrangement of curves which satisfy the condition that each pair of curves intersect at most s times for a given constant s, is also important in both theory and applications[9, 13]. When $s = 1$ and the curves are x-monotone and unbounded, such an arrangement is known as an arrangement of pseudo-lines, to which many results on an arrangement of lines generalizes. For example, the complexity of the k-level of an arrangement of n pseudo-lines is known to be $O(\sqrt{k}n)$ [15, 4].

We focus on the case $s = 2$ in this paper. A familiar example of such an arrangement is that of axis-parallel parabolas in which two curves intersect at most twice. An arrangement of axis-parallel parabolas is used in dynamic computational geometry, since it shows the transition of L_2 distances among a set of linearly moving points. The complexity of the lower envelope and the k-level of an arrangement of parabolas gives the number of combinatorial changes on the nearest pair and the nearest k pairs, respectively [1]. Also, the complexity of topological change on the L_2 minimum spanning tree can be formulated into a problem on an arrangement of parabolas [10].

More generally, we consider an arrangement of unbounded x-monotone Jordan curves intersecting each other at most twice. Although x-monotone condition is not necessary for deriving the results except in Section 5, we assume it for simplicity in this paper. Such an arrangement is called an arrangement of 2-intersecting curves in the literature; however, for convenience's sake, we call it an arrangement of *pseudo-parabolas.*

It is often more difficult to analyze the complexity of an arrangement of curves than an arrangement of lines or pseudo-lines. For example, the only upper bound previously known on the k-level complexity of an arrangement of parabolas is $O(kn)$, which is the same as the bound for $\leq k$-levels[13].

The aim of this paper is to link the complexity of an arrangement of pseudo-parabolas to that of an arrangement of pseudo-lines. Our approach is to split pseudo-parabolas by cut points, generating an arrangement of ***pseudo-segments*** in which each pair of pseudo-segments intersect at most once. For example, the arrangements of Figure 1 can be made into an arrangement of pseudo-segments by giving seven cuts. We call the minimum of cuts to make an arrangement Γ into arrangement of pseudo-segments the *cutting number* of Γ.

Our main results are the following two theorems:

Theorem 1 *There exits an arrangement of axis parallel parabolas whose cutting number is* $\Omega(n^{4/3})$.

Theorem 2 *The cutting number of an arrangement of pseudo-parabolas is* $O(n^{5/3})$.

We also give the same bounds for the cutting number of an arrangement of circles.

The lower bound of Theorem 1 is derived from the Edelsbrunner-Welzl's lower bound example [5] on the complexity of n cells in an arrangement of lines.

The upper bound of Theorem 2 is derived from an ineauality of Lovász's used in the proof of his fractional covering theorem [11], combined with extremal graph theory [2] and a probabilistic method [3, 13]. A greedy algorithm outputs cuts attaining this upper bound.

Combining Theorem 2 with the known upper bound on the level complexity of an arrangement of pseudo-lines, we derive a non-trivial $O(n^{23/12})$ upper bound on the complexity of a level of an arrangement of pseudo-parabolas. The technique used here is such that any improved upper bound for pseudo-lines will lead to an improved upper bound for pseudo-parabolas. Thus, Theorem 2 establishes an important link between the complexities of arrangements of these two types.

Also, we improve some bounds on problems in matroid theory and dynamic computational geometry. For instance, we improve the upper bound in [10] of the number of transitions of L_2 Euclidean minimum spanning trees of linearly moving points by a factor of $n^{1/12}$.

2 Preliminary

Let Γ be an arrangement of pseudo-parabolas. The arrangement makes a subdivision of the plane into faces. We use the terms *cell*, *edge*, and *vertex* for 2, 1, and 0 dimensional faces, respectively. A *lens* of the arrangement is the boundary of a closed region bounded by two pseudo-parabolas; we say that these pseudo-parabolas form a lens.

The boundaries of the shaded regions in Figure 2 are lenses. We say a lens is a 1-lens if no curve cross the lens. Consequently, a 1-lens consists of exactly two edges of the arrangement. There exists no 1-lens in Figure 2.

We have the following lemma, which will be used in the proof of the lower bound:

Lemma 3 *The cutting number of Γ is not less than the number of 1-lenses.*

Proof: One of the two edges in each 1-lens must be cut, and an edge is contained in at most one 1-lens. Thus, the lemma follows. □

We define a hypergraph $H(\Gamma)$, whose node set is the set of edges of the arrangement Γ. A set of nodes of $H(\Gamma)$ forms a hyperedge if and only if its corresponding set of edges of the arrangement forms a lens.

The node covering of a hypergraph H is a subset of the node set of H such that every hyperedge contains at least one node of the set.

The node covering with minimum size (number of nodes) is called the minimum covering. The size of the minimum covering is called the covering number.

The following is a key lemma for our upper bound of the cutting number:

Lemma 4 *The cutting number of Γ equals the covering number of $H(\Gamma)$.*

Proof: Given a minimum covering C of $H(\Gamma)$, we cut all edges of Γ associated with nodes in C. Then all lens are cut, that means all pair of curve segments after the cut intersect at most once. On the contrary, given a minimum cut of Γ, consider the collection of the edges cut. Then associated node set of $H(\Gamma)$ is a covering. □

3 Lower bound

We give a proof of Theorem 1 in this section. We use the following well-known fact:

Theorem 5 *[5] The total complexity of m cells in an arrangement of lines is $\Omega(m^{2/3}n^{2/3})$ if $m > n^{1/2}$.*

In particular, the total complexity of n cells is $\Omega(n^{4/3})$. The total sum of the numbers of edges in the upper chains of n cells is also $\Omega(n^{4/3})$. We first show the same lower bound for the arrangement of pseudo-parabolas.

Proposition 6 *There exists an arrangement of pseudo-parabolas whose cutting number is $\Omega(n^{4/3})$.*

Proof: Let $\mathcal{A}$ be an arrangement of n lines, and R be a cell of $\mathcal{A}$. We choose n cells $R_1, R_2,..,R_n$ such that the total complexity of upper chains of those cells is $\Omega(n^{4/3})$.

For each cell R, we can draw an x-monotone connected curve segment $\gamma(R)$ in the closure of R, such that $\gamma(R)$ is tangent to every edge in the upper chain of R. We make a curve $\tilde{\gamma}(R)$ which separates the plane by adding a vertical downwards ray to each endpoint of $\gamma(R)$.

Consider the arrangement $\mathcal{B}$ of the union of the set of lines in $\mathcal{A}$ and the set of curves $\{\tilde{\gamma}(R_i) : i = 1, 2, .., n\}$ The number of touching points between curves and lines in $\mathcal{B}$ is $\Omega(n^{4/3})$. Now, we perturb $\mathcal{B}$ by slightly translating all curves $\tilde{\gamma}(R_i)$ $i = 1, 2, .., n$ upwards. Then, the arrangement has $\Omega(n^{4/3})$ 1-lenses. It is not difficult to see that each pair of curves (and lines) in this arrangement intersects at most twice. Thus, the proposition follows from Lemma 3. $\square$

Next, we show that the lower bound can be attained in an arrangement of parabolas. To show the lower bound example of n cells in an arrangement, Edelsbrunner and Welzl [5] construct a highly degenerate arrangement $\mathcal{A}'$ which has a set V of n vertices with total degree $\Omega(n^{4/3})$. Also, due to that construction, we can assume that each line of the arrangement has a positive slope which is not larger than $\pi/4$.

We consider a very skinny axis-parallel parabola γ, and draw a copy $\gamma(v)$ which has its peak at v for each v in V. Now, we translate each line, so that the degeneracy of $\mathcal{A}'$ is resolved, and each line through a point v in V is translated so that it is tangent to $\gamma(v)$. Since the point of contact of a line to a given parabola is determined by the slope of the line, the translation is unique for a line with a given slope. Hence, the translation does not generate global inconsistency.

We consider the union of $\mathcal{A}'$ and the translated parabolas. Then, the number of vertices in this arrangement at which two curves are tangent is $\Omega(n^{4/3})$. Thus, by perturbing this arrangement, we obtain a lower bound example proving Theorem 1.

4 Upper bound

It suffices to give an upper bound for the covering number of hypergraph $H(\Gamma)$. We recall some notations on hypergraphs. The degree $d(x)$ of a node x of a hypergraph is the number of hyperedges containing x. The maximum degree in a hypergraph H is denoted by $d(H)$.

A k-matching of H is a collection $\mathcal{M}$ of hyperedges (the same edge may occur more than once) such that each node belongs to at most k of them. A k-matching is simple if no edge occurs in it more than once. The maximum number of hyperedges in a simple k matching is denoted by $\nu_k(H)$ (this is denoted by $\tilde{\nu}_k(H)$ in Lovász [11]). Note that $\nu_{d(H)}(H)$ is the number of hyperedges in H. We remove the argument H from functions d and ν, if no confusion arises. A greedy algorithm for computing a covering is the following:

1: Find the node of maximum degree;

2: Insert the node in the covering,

and remove the node and all edges containing it from H;

3: If all edges are covered, Exit; Else GOTO 1;

Lovász [11] shows that the greedy algorithm achieves a covering of size at most $\log d(H) + 1$ times the covering number of H. The following is the key inequality in his proof. Let t be the number of covering of H obtained by a greedy algorithm. Then,

$$t \leq \frac{\nu_1}{1 \cdot 2} + \frac{\nu_2}{2 \cdot 3} + \cdots + \frac{\nu_{d-1}}{(d-1) \cdot d} + \frac{\nu_d}{d} \quad (1)$$

Consequently, the minimum covering number of H is also bounded by the righthand side of (1). Therefore, we want to estimate $\nu_k(H(\Gamma))$ for $k = 1, 2, ..d$.

Suppose we have a simple 1-matching $\mathcal{M}$ of $H(\Gamma)$ of size M. Recall that a hyperedge in $H(\Gamma)$ is a lens in Γ.

We define a bipartite graph $G(\mathcal{M})$. The vertex set is $S_1 \cup S_2$, where S_1 and S_2 are disjoint, $|S_1| = |S_2| = |\Gamma| = n$, with associated bijections $\gamma_1 : S_1 \to \Gamma$ and $\gamma_2 : S_2 \to \Gamma$.

We draw an edge between a node u of S_1 and v of S_2 if and only if the associated curves $\gamma_1(u)$ and $\gamma_2(v)$ form a lens which is associated with a hyperedge in $\mathcal{M}$, and $\gamma_1(u)$ is above $\gamma_2(v)$ within the lens. Here, a curve γ is *above* another curve μ within their lens L if a vertical downward ray from a point on $\gamma \cup L$ intersects μ. (Note: here, we use x-monotonicity. We need a more complicated definition without the assumption).

By definition, $G(\mathcal{M})$ has $2n$ vertices. It is clear that the number of edges in $G(\mathcal{M})$ is the size of the matching $\mathcal{M}$.

First, we consider the number $\nu_1(H(\Gamma))$ of 1-matchings.

Lemma 7 *Suppose $\mathcal{M}$ is a simple 1-matching of $H(\Gamma)$. Then, $G(\mathcal{M})$ does not contain $K_{3,4}$ as a subgraph.*

Proof: Assume that $G(\mathcal{M})$ contains a copy of $K_{3,4}$. Then, we have three curves C_1, C_2, C_3 and four curves D_1, D_2, D_3, D_4 such that each pair (C_i, D_j) makes a lens, in which C_i is above D_j, for $1 \leq i \leq 3, 1 \leq j \leq 4$. Furthermore, because $\mathcal{M}$ is a 1-matching, no edge in the arrangement constructed from these seven curves is contained in more than one such lense. Let $A(C)$ denote the arrangement of $\{C_1, C_2, C_3\}$ and $A(D)$ that of $\{D_1, D_2, D_3, D_4\}$. If an edge e is located on D_1, and e is below two curves C_1 and C_2, e must be on both of two lenses (C_1, D_1) and (C_2, D_1). This means that the two arrangements $A(C)$ and $A(D)$ intersect each other only at points that are on the upper envelope of $A(C)$. Similary, those intersection points must be on the lower envelope of $A(D)$ at the same time. Since there must be 12 lenses, the number of intersections must be at least 24. However, the upper envelope of $A(C)$ has at most 5 edges, and the lower envelope of $A(C)$ has at most 7 edges. Because each curve intersects at most twice, the number of intersecting points cannot exceed 22, which is a contradiction. □

Remark. We can also show that $G(\mathcal{M})$ does not contain $K_{3,3}$ with more careful analysis.

We use the following result in extremal graph theory, which can be found in Bollobas [2] in a more general form (page 73, Lemma 7).

Lemma 8 *Let $G(n,n)$ be a graph without a $K_{s,t}$ subgraph. Suppose $G(n,n)$ contains $m = yn$ edges. Then, $n\binom{y}{s} \leq (t-1)\binom{n}{s}$.*

Theorem 9 *$G(\mathcal{M})$ contains $O(n^{5/3})$ edges. Hence, $\nu_1(H(\Gamma)) = O(n^{5/3})$.*

Proof: We substitute $s = 3$ and $t = 4$ in the above lemma, and obtain $y = O(n^{2/3})$. □

Next, we consider $\nu_k(H(\Gamma))$, for $k \leq \sqrt{n}$. We apply the probabilistic argument similar to the one Sharir [13] used for analyzing the complexity of $\leq k$ level of curves.

Suppose we have a simple k-matching $\mathcal{M}$, and associated set $\mathcal{L}(\mathcal{M})$ of lenses. Assume $\mathcal{M}$ has ν_k hyperedges (i.e. $\mathcal{L}(\mathcal{M})$ has ν_k lenses).

For each lens L bounded by two curves C_1 and C_2, its edge is called extremal if it contains one of the intersection points of C_1 and C_2. Obviously, there are at most four extremal edges associated with L.

Now, we choose a random sample Y of n/k curves from Γ. We say a lens L is *near 1-lens* of the sample if (1) L is a lens consisting of two curves of Y and (2) for each extremal edge e of L, L is the only lens which consists of two curves of Y and contains e.

Let us consider the set of near 1-lens, and consider the associated matching $\mathcal{M}_0$ of $H(Y)$.

Lemma 10 *$\mathcal{M}_0$ is a 1-matching of $H(Y)$.*

Proof: Suppose an edge e of the arrangement Y is contained in two near 1-lenses L and L'. Let C be the curve on which e is located. Then, L and L' contain intervals I and I' of C. Both intervals must contain e; thus, at least one endpoint of either I or I' must be contained in the other interval. W.l.o.g., we assume an endpoint of I is contained in I'. This means that an extremal edge of L is contained in L', which contradicts the definition of near 1-lens. □

Now, analyze the expected number of near 1-lenses. A lens L of $\mathcal{L}(\mathcal{M})$ becomes a near 1-lens if (1) both of its bounding curves are in the sample and (2) no other curve contributing a lens containing an extremal edge of L is in the sample. Since $\mathcal{M}$ is a k-matching, the number of such curves in (2) is at most k for each extremal edge.

Thus, the probability that L becomes a near 1-lens is at least $k^{-2}(1-1/k)^{4k}$, hence the expected number of near 1-lenses is at least $k^{-2}(1-1/k)^{4k}\nu_k$. Note that $(1-1/k)^{4k} > 1/256$.

Since the number of 1-matchings in the sample is $O((n/k)^{5/3})$, $\nu_k = O(k^2(n/k)^{5/3}) = O(n^{5/3}k^{1/3})$.

Now, let us compute the righthand side of the Lovázs's inequality (1).

$$\frac{\nu_1}{1\cdot 2} + \frac{\nu_2}{2\cdot 3} + \cdots + \frac{\nu_{d-1}}{(d-1)\cdot d} + \frac{\nu_d}{d}$$
$$= O(n^{5/3}[\sum_{k=1}^{\sqrt{n}} \frac{k^{1/3}}{k(k+1)}] + \sum_{k=\sqrt{n}}^{n} \frac{n^2}{k(k+1)})$$
$$= O(n^{5/3}).$$

This proves Theorem 2.

5 Applications

Level complexity

Let Γ be an arrangement of n pseudo-parabolas. In this section, we need the assumption that each curve is x monotone and unbounded. The *level* of an edge e of the arrangement Γ is the number of edges which intersect with the vertical half line downward from an internal point on e. Level is well-defined since the above number is independent of the choice of the internal point of e. It is well-known that the union of all edges with a given level k is a connected chain of curve and separates the plane. This chain is called the k-level of Γ. The complexity of k-level of Γ is the number of edges whose level is k.

Theorem 11 *The complexity of k-level of Γ is $O(n^{23/12})$.*

Proof: Without loss of generality, we assume that the arrangement is simple, that means no three curves intersect at a point. We consider the set P of cutting points of Γ into pseudo-line arrangements. We know that $m = |P| = O(n^{5/3})$. We subdivide the plane into $m+1$ slabs with m vertical lines through points of P. Inside a slab, the arrangement can be considered as an arrangement of pseudo-lines.

Let X_i be the number of vertices of the arrangement located in the i-th slab S_i. Then, $\sum_{i=1}^{m+1} X_i = O(n^2)$.

Suppose that exactly n_i curves contribute to k-level in S_i. The k-level of Γ inside S_i is a level of these n_i curves.

Suppose a curve γ is on k'-level at the left end of the slab S_i, and contributes to k-level in S_i. Then, γ must have at least $|k-k'|$ vertices on it in S_i. Therefore, the arrangement must have at least $n_i^2/2$ vertices in S_i. This means that $n_i = O(\sqrt{X_i})$.

The complexity of a level of n_i pseudo-lines is $O(n_i\sqrt{n_i})$, which is $O(X_i^{3/4})$.

Thus, the complexity of the k-level of Γ is $O(\sum_{i=1}^{m+1} X_i^{3/4}) = O(m(n^2/m)^{3/4})$. Since $m = O(n^{5/3})$, we obtain $O(n^{23/12})$. □

Note that this result improves the known bound of $O(kn)$ [13] when $k > n^{11/12}$. Also note that the bound would be automatically improved further, if we had a better bound either on the level complexity of an arrangement of pseudo-lines or on the cutting number of pseudo parabolas.

Transitions of minimum matroid base and MST

Let E be a finite set and $\mathcal{B}$ a family of subsets of E. The pair $(E, \mathcal{B})$ is called a *matroid* $M(E, \mathcal{B})$, and the elements of $\mathcal{B}$ are the *bases* of $M(E, \mathcal{B})$, if the following two axioms hold [14]:

(A1) For any $B, C \subset E$ with $B \neq C$, if $B \in \mathcal{B}$ and $C \subset B$, $C \notin \mathcal{B}$.

(A2) For any $B, B' \subset \mathcal{B}$ with $B \neq B'$ and for any $e \in B - B'$, there exists $e' \in B' - B$ such that $(B - \{e\}) \cup \{e'\} \in \mathcal{B}$.

For instance, let $\mathcal{T}$ be a set of spanning trees in an undirected connected graph $G = (V, E)$; then $(E, \mathcal{T})$ forms a matroid and $\mathcal{T}$ is a set of bases [14].

The number $|B|$ of elements of a base $B \in \mathcal{B}$ is independent of the choice of B [14], and is denoted by p. Let $m = |E|$, and assume the elements of E to be indexed from 1 through m. We assume that each element i has a real-valued weight $w_i(t)$ that is a function in the parameter t. The minimum (resp. maximum) weight base is the one in which the sum of weights of elements is minimum (resp. maximum).

If the weight functions of two elements have constant number of intersections, we have an

$O(m^2)$ trivial upper bound on the number of transitions of the minimum (resp. maximum) weight base of $M(E, \mathcal{B})$. If $w_i(t)$ is linear, this was improved to $O(m \min\{\sqrt{p}, \sqrt{m-p}\})$ by [6, 8].

We have the following theorem, which can be proved similarly to Theorem 11:

Theorem 12 *When all $w_i(t)$ are quadratic in t, the number of transitions is $O(m^{23/12})$*

Corollary 13 *Let G be a graph with m edges, and each edge has a weight function which is quadratic in a parameter t. Then, the number of transitions in its minimum spanning tree is $O(m^{23/12})$.*

Theorem 14 *Let S be a set of n linearly moving points in d dimensional space, where d is a constant. Then, the transition of Euclidean minimum spanning tree is $O(n^{35/12}2^{\alpha(n)})$, where α is the inverse Ackerman function.*

Proof: See [10] for the L_1-distance analogue of this theorem, and replace the number of transitions of linear-weighted MST used there by that of quadratic-weighted MST shown above. □

This improves the known $O(n^3 2^{\alpha(n)})$ bound [10] by a factor of $n^{1/12}$.

6 Related topics

Competitive ratio of the greedy algorithm

The upper bound of the cutting number is obtained by analyzing the greedy algorithm. However, one may suspect that the real cutting number is much smaller than the cutting number output by the greedy algorithm. However, Lovász [11] gave the following performance ratio of the greedy algorithm:

Theorem 15 *The competitive ratio of the greedy algorithm is $O(\log d)$ for the minimum covering problem, where d is the largest degree of any node of the hypergraph.*

Thus, the ratio of the cutting number of the greedy algorithm to the optimal one is $O(\log n)$.

Arrangement of circles

We are given an arrangement of n circles. Although a pair of circles intersects at most twice, a circle is not a simple curve. However, we can cut each circle with its horizontal diameter, and divide it into an upper half-circle and lower half-circle. We can connect two vertical downwards (resp. upwards) rays to an upper (resp. lower) half-circle at its two endpoints, and obtain a simple curve separating the plane. It is easy to see that every pair of curves intersects at most twice.

Thus, we have a family of pseudo-parabolas. We can now apply our upper bound results in the previous section. Also, if we are permitted to consider a line as a circle with infinite diameter, we can deform our lower bound example so that it holds for circles.

Thus, we have the following:

Theorem 16 *Using $O(n^{5/3})$ cuts, an arrangement of circles can be transformed to an arrangement of pseudo-segments. There exists an example for which $\Omega(n^{4/3})$ cuts are required.*

Cutting a $2t$ intersecting family

It is desired to extend the upper bound on cutting number to that for an arrangement of curves intersecting t times each other. Unfortunately, the technique employed in this paper does not work for cutting such an arrangement into an arrangement of pseudo-segments. We only have the following weak result (we omit the proof):

Theorem 17 *Given an arrangement of curves in which every pair of curves intersects at most $2t$ times, we can cut it at $O(n^{2-1/\beta(t)})$ points to make it an arrangement of curve segments, in which every pair of segments intersects at most $2t-1$ times. Here, $\beta(t)$ is the minimum positive number y satisfying $y^2 \geq 4t\lambda_{2t}(y)$, where $\lambda_{2t}(y)$ is the length of Davenport-Schinzel sequence of degree $2t$ on y characters.*

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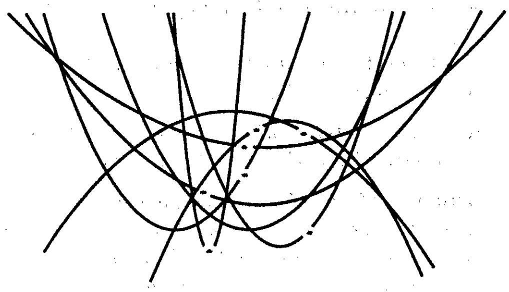

Figure 1: Cut points

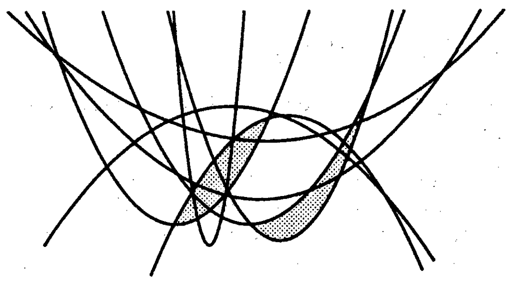

Figure 2: Lenses