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A visibility problem: optimal postings of monitors

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Abstract

We consider the problem of posting monitors to watch the boundaries of objects in an enclosure. Algorithms are developed to compute optimal posting, which is a placement of the minimum number of monitors so that every boundary point of the objects in the enclosure is visible by some monitor. Specifically, we present algorithms for the four cases: (1) a circle in a circle, (2) two circles in a circle, (3) a circle in a convex polygon, and (4) two convex polygons in a convex polygon.

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アブストラクト

本文では、敷地内の物体の境界を監視するモニターテレビの配置問題について考察する。敷地内の物体の境界上の点がすべてどれかのモニターテレビで見えるように最小個数のモニターテレビを配置する方法を求めるアルゴリズムを開発した。特に、次の4つの場合についてアルゴリズムを述べる。(1) 円の中に円がある場合、(2) 1つの円の中に2つの円がある場合、(3) 凸多角形の中に円がある場合、(4) 凸多角形の中に2つの凸多角形がある場合。

1 Introduction

The art gallery theorem and its variations have received much attention [2,3,5,6,7,8]. We examine the following variations: Suppose that there is a collection of mutually disjoint finite closed regions(the objects) properly contained in a finite closed region (the enclosure). A proper posting is a positioning of a number of stationary monitors within the enclosure but outside the objects so that every point of the boundary of each object is seen by some monitor. An optimal posting is a proper posting with the minimum number of monitors.

In [4] the following results were presented:

- When there is one convex n-sided polygonal object inside a convex m-sided polygonal enclosure, an optimal posting may be computed in linear time $O(n+m)$. Also for a convex n-sided polygon in a non-convex m-sided polygon, an optimal posting may be computed in time $O(nm)$.
- The problem of finding an optimal posting for a collection of convex objects in a convex enclosure is NP-*hard*.
- The problem of computing an optimal posting for a single non-convex polygon in a convex enclosure is NP-*hard*.
- The 3-dimensional problem of computing an optimal posting for a convex polyhedron in a convex polyhedral enclosure is also NP-*hard*.

In this paper we consider the problems of computing optimal posting for the following situations:

1. single circular object in a circular enclosure.
2. two circular objects in a circular enclosure.
3. single circle in a convex polygon.
4. two convex polygons in a convex polygon.

We use the following notation throughout the paper: For points a and b, ab denotes the line segment connecting a and b, $\overline{ab}$ the line passing through a and b, $a^{\rightarrow}b$ the halfline from a through b in the sense from a to b, and $a^{\leftarrow}b$ the halfline from b through a in the sense from b to a. The enclosure is denoted by E, and an object is denoted by R. The boundaries of E and R are denoted by ∂E and ∂R, respectively. Their interiors are $E^o = E - \partial E$ and $R^o = R - \partial R$, respectively. The open set $E^o - (\cup R)$ is referred as the open area of E. For an n-sided convex polygonal enclosure E, its vertices are denoted $u_0, u_1, \ldots, u_{n-1}$ listed counterclockwise. For convex polygonal objects R and S, we list their vertices counterclockwise as $v_0, v_1, \ldots, v_{m-1}$ and $w_0, w_1, \ldots, w_{l-1}$, respectively. When the enclosure and the objects are circles, then their centers are denoted by c_E, c_R and c_S, and their radii by r_E, r_R and r_S, respectively. We use d_i, e_i, f_i to denote edges u_iu_{i+1}, v_iv_{i+1} and w_iw_{i+1}, where index $i+1$ is <u>mod</u> m, n and l, respectively. For two points p and q on ∂E, $\partial E(p,q)$ is the subset of ∂E from p to q traversed counterclockwise. A posting is a set $P = \{p_1, p_2, \ldots, p_t\}$ of positions of monitor placement, listed counterclockwise in the given order.

2 Objects in a circular enclosure

We investigate the problems of finding optimal posting for a circle in a circle and two circles in a circular enclosure.

2.1 single circular object

Consider the situation where single circular object in a circular enclosure need be monitored. We may assume that in an optimal posting, every monitor is placed on the boundary ∂E of the enclosure E. For, suppose that an optimal posting P has a monitor placed at p in the interior E^o of E. Let p' be the point of intersection of $\overline{c_R p}$ and ∂E with p on $c_R p'$. Move the monitor at p to p'. Then the new posting is also proper and thus is optimal. In this way every monitor in P that are placed in E^o may be moved to ∂E and the new posting is still proper and optimal.

We first make a simple observation. For an n-sided convex polygonal enclosure, the number of monitors in an optimal posting is bounded by n. However, if the enclosure is a circle, the number of an optimal posting is unbounded. For instance, suppose that the object R and the enclosure E are concentric circles of radii r_R and r_E, where $n(\geq 3)$ is an integer and $r_R/r_E = \cos(\pi/n)$. Then the regular n-gon inscribed in E circumscribes R, and the set of the n vertices of the polygon is an optimal posting for the case.

The idea of the algorithm is due to the *Poncelet theorem*[1]: Given two circles C and C' with C inside, construct a sequence of points p_i $(i \geq 0)$ on C', beginning with a given $p_0 \in C'$, such that $p_i p_{i+1}$ is tangent to C. Then either there exists n such that $p_n = p_0$ for all $p_0 \in C'$ or $p_i \neq p_0$ for all $p_0 \in C'$ and $i > 0$. Figure 1 shows an instance of an optimal posting for a circle in a circle.

<u>*algorithm*</u> C,C-post$((c_R, r_R), (c_E, r_E))$
{ find p_0 and p_1 on ∂E such that $p_0 p_1$ is tangent to R;
  initialize the posting P to $\{p_0, p_1\}$; set flag *done* to *false*; set index i to 1;
  <u>*while*</u> not *done* <u>*do*</u>
  { let p_{i+1} $(\neq p_{i-1})$ be the point on ∂E such that $p_i p_{i+1}$ is tangent to R;
    <u>*if*</u> $p_i p_{i+1}$ intersects $p_0 p_1$ <u>*then*</u> set *done* to *true*
    <u>*else*</u> { update P to $P \cup \{p_{i+1}\}$; set i to $i+1$ }
  }
}

Theorem **1** : Algorithm C,C-post *finds an optimal posting in time $O(t)$ time, where t is the number of monitors in the optimal posting.*

2.2 two circular objects

We now consider the situation where two circular objects R and S are to be monitored inside a circular enclosure E. As in the case of a circle in a circle, the minimum number of monitors required is not bounded. Even though not obvious at present, it is always possible to obtain an optimal posting by placing all monitors only on the boundary ∂E of the enclosure.

Lemma **2** : *To monitor two circular objects in a circular enclosure, there is always an optimal posting with all monitors on the boundary ∂E.*

The algorithm to obtain an optimal posting for the problem is a modification of *algorithm C, C-post* for a circle in a circle. Figure 2 shows an optimal posting obtained by the algorithm for two circular objects in a circular enclosure.

<u>*algorithm*</u> 2C,C-post$((c_R, r_R), (c_S, r_S), (c_E, r_E))$
{ find p_0 and p_1 on ∂E such that $p_0 p_1$ is tangent to both R and S with
  each on the opposite side of $p_0 p_1$;
  initialize the posting P to $\{p_0, p_1\}$; set flag *done* to *false*; set indes i to 1;

while not *done* *do*
{ let p_{i+1} ($\neq p_{i-1}$) be the point on ∂E such that $p_i p_{i+1}$ is tangent to R;
  if $p_i p_{i+1}$ intersects $p_0 p_1$ *then* set *done* to *true*
  else { update P to $P \cup \{p_{i+1}\}$; set i to $i+1$ }
}
let p_{i+1} ($\neq p_1$) be the point on ∂E such that $p_0 p_{i+1}$ is tangent to S;
update P to $P \cup \{p_{i+1}\}$; increment i by 1;
while not *done* *do*
{ let p_{i+1} ($\neq p_{i-1}, p_0$) be the point on ∂E such that $p_i p_{i+1}$ is tangent to S;
  if $p_i p_{i+1}$ intersects $p_0 p_1$ *then* set *done* to *true*
  else { update P to $P \cup \{p_{i+1}\}$; set i to $i+1$ }
}

Theorem **3** *:* Algorithm 2C,C-post *finds an optimal posting in time* $O(t)$, *where* t *is the number of monitors in the optimal posting.*

3 objects in a convex polygonal enclosure

Since the case of a convex polygon in a convex polygon was studied in [4], we investigate the cases of a circle in a convex polygon and two convex polygons in a convex polygon.

3.1 single circular object

Obviously as for the case of a circle in a circle, we may assume that in an optimal posting for this stuation, every monitor may also be placed on the boundary of the enclosure. Unlike in the former case, however, the stratege of placing the first monitor at any point on ∂E and then successively placing the monitors as far away from the previous one as possible does not always work for this case. For the instance shown in Figure 3, regardless where the first monitor is placed, the strategy finds an optimal posting. For the case shown in Figure 4, however, the strategy places different number of monitors depending on the position of the first moniotor. In particular, if the first monitor is placed at p_0, then only four monitors are required to watch the circle, but if it is started with a monitor at q_0, five monitors are required. Fortunately, the difference in the number of monitors required depending on the initial position is at most one.

Below we present a procedure which, if terminates, produces an optimal posting. In such cases, in addition to finding an optimal posting as the algorithm, the procedure confirms its optimality. However, it falls into an infinite loop if there exists an inscribing and circumscribing polygon.

procedure C,P-post$((c_R, r_R), E)$
{ set *startpoint* to u_0; set flag *completed* to *false*;
  while not *completed* *do*
  { set p_0 to *startpoint*; initialize the posting P to $\{p_0\}$;
    set flag *done* to *false*; set index i to 0;
    while not *done* *do*
    { let p_{i+1} ($\neq p_{i-1}$) be the point on ∂E such that $p_i p_{i+1}$ is tangent to R;
      if $p_i p_{i+1}$ intersects $p_0 p_1$ *then* set *done* to *true*
      else { update P to $P \cup \{p_{i+1}\}$; set i to $i+1$ }
    }
    if u_0 is a point of $\partial E(p_0, p_i)$ *then* set *completed* to *true*
    else reset *startpoint* to p_i;
  }

}

Theorem **4** : *If it terminates, the* procedure C,P-post *produces an optimal posting.*

The procedure may not terminate if there exits a convex polygon that is inscribed in E and circumscribes R. For, as the last monitor p_t of the posting obtained in *while*-loop approaches a vertex of a convex polygon that is inscribed in E and circumscribes R, p_0 and p_t gets ever closer in successive execution of *while*-loops while p_0 never actually reaching the vertex. Even if there is no such polygon, p_0 and p_t may get ever closer executing unbounded number of the *while*-loop before they start separating again.

3.2 two convex polygonal objects

Here we consider the problem of monitoring two convex polygonal objects placed in a convex polygonal enclosure. Unlike the case of two circles in a circle, it may be necessry to place a monitor in E^o, the interior of the enclosure E. Figure 5 shows an instance where any optimal posting must have a monitor placed in E^o. It will be shown, however, that an optimal posting may need at most one monitior in E^o. When an optimal posting needs a monitor in E^o, it may be placed only at the intersection of the extensions of an edge of R and an edge of S. Since there is no result that determines the two edges the intersection of whose extensions is the position of the monitor in E^o, the algorithm presented here is an exhaustive one.

Lemma **5** : *If in an optimal posting P, a monitor p must be placed in E^o, then there is a line passing p that separates the two polygonal objects R and S.*

Let p be a monitor in ∂E, and let $e_i, e_{i+1}, \ldots, e_j$ be the edges of R listed counterclockwise that are visible from p and $f_h, f_{h+1}, \ldots, f_k$ the deges of S visible from p. The four lines $\overline{e_i}$, $\overline{e_j}$, $\overline{f_h}$, and $\overline{f_k}$ produce a 4-sided polygon. All edges visible from p are visible from any point in the polygon, which is referred to as the domain of the monitor p.

Lemma **6** *Let p be a monitor in a proper posting P which is placed in E^o. If its domain is not a proper subset of E^o, then p may be relocated to a point on ∂E and P is still proper.*

We will now see that in any proper posting, no more than one monitor is required to be placed in E^o.

Lemma **7** : *An optimal posting need not have more than one monitor placed in E^o.*

Suppose p is the only monitor in an optimal posting P which is placed in E^o. Then P remains an optimal posting if p is relocated to any point in its domain. Thus, we may assume that the monitor in E^o in an optimal posting is placed at the intersection of the extensions of an edge of R and an edge of S.

Below we present an informal description of an algorithm that computes an optimal posting. To do so, we need additional notations. Recall that d_i, e_i, f_i denote edges u_iu_{i+1}, v_iv_{i+1} and w_iw_{i+1}, where index $i+1$ is $\underline{mod}$ m, n and l, respectively. For edge $d_i = u_iu_{i+1}$, $\overline{d_i}$, $d_i^{\rightarrow}$, and $d_i^{\leftarrow}$ represent $\overline{u_iu_{i+1}}$, $u_i^{\rightarrow}u_{i+1}$ and $u_i^{\leftarrow}u_{i+1}$, respectively. Similar notation is used for edges e_i and f_i. Let L be a line that separates R and S to partition E, and let E_R and E_S denote the partitions of E including R and S, respectively. Now let r_i denote the point of intersection of $e_i^{\rightarrow}$ and ∂E_R and s_i the intersection point of $f_i^{\rightarrow}$ and ∂E_S. While computing a proper posting, if a monitor is placed at r_i (s_i), then the next monitor is placed at $r_j(s_j)$ such that edges $e_i, e_{i+1}, \ldots, e_{j-1}(e_i, e_{i+1}, \ldots, e_{j-1})$ are all visible from $r_i(s_i)$ but $e_j(f_j)$ is not visible.

algorithm 2P,P-post(R, S, E)
{ find a line L that separates R and S;
compute the proper postings P_R for R in E_R;
compute the proper postings P_S for S in E_S;
obtain an optimal posting P_1 for R and S in E by combining P_Rs and P_S and moving the monitors on L to one of the intersection points of L and ∂E;
for each intersection of $\overline{e_i}$ and $\overline{f_i}$ whose domain is properly contained in E *do*
{ let p_0 denote the intersection point;
find a line L that passes p_0 and separates R and S;
obtain a proper posting P_2 with p_0 and other monitors to watch the edges of R and S that are not invisible from p_0
}
set P to P_1 or P_2 with less number of monitors
}

Theorem **8** : Algorithm 2P,P-post *finds an optimal posting for two convex polygons in a convex polygon in time* $O(ml(n+m+l))$, *where* n, m *and* l *are the number of the vertices of* E, R *and* S, *respectively.*

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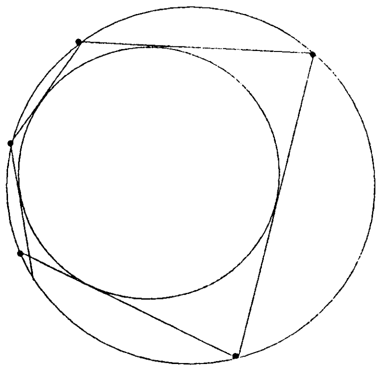

Figure 1: An optimal posting for a circle in a circle

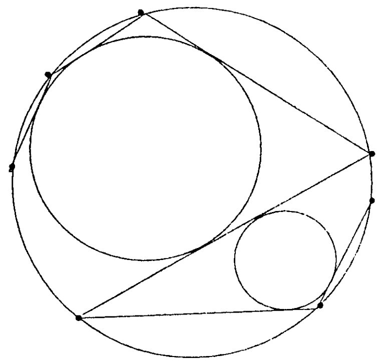

Figure 2: An optimal posting for two circles in a circle

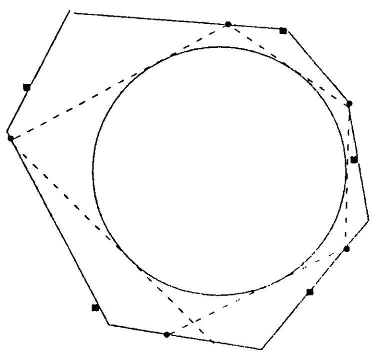

Figure 3: Optimal postings from any point

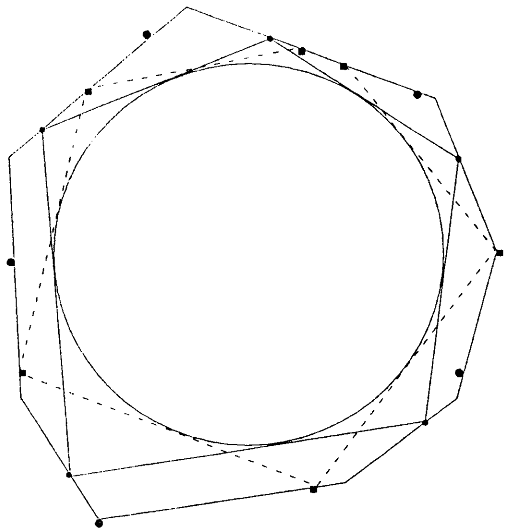

Figure 4: Postings for a circle in a polygon

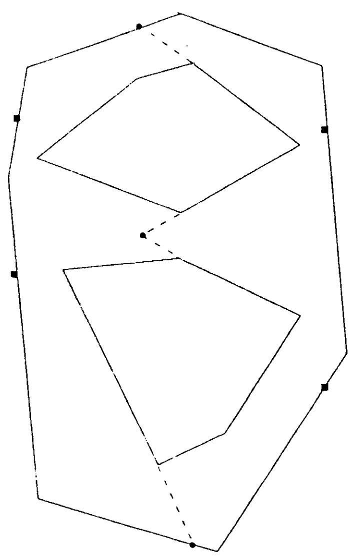

Figure 5: Postings for two polygons in a polygon