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Kautz ダイグラフの同型分解と弧素全域木

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概要： 本論文では Kautz ダイグラフの分解や弧素全域木といった構造的性質を調べる。そして、完全対称ダイグラフの同型分解を適用することによって Kautz ダイグラフの同型分解を与える。又、Kautz ダイグラフの巡回頂点を根とする弧素全域木を構成する。

Isomorphic decompositions and arc-disjoint spanning trees of Kautz digraphs

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Abstract. In this paper, we study the structural properties of the Kautz digraph such as decomposition and arc-disjoint spanning trees. We present several isomorphic decompositions of the Kautz digraph applying isomorphic decompositions of the complete symmetric digraph. Also we construct arc-disjoint spanning trees rooted at a cyclical vertex of the Kautz digraph.

1 Introduction

In the design of massive parallel computers, the choice of the interconnection network is very important. The topology of the interconnection network is one of the critical factors which determine the performance of the parallel computer.

So far, a lot of interconnection networks have been proposed for massive parallel computers. Among them, the de Bruijn and Kautz networks have been noticed because of their good properties such as small diameter, high connectivity, bounded degree and so on. In this paper, we study their structural properties such as decompositions and arc-disjoint spanning trees.

Let G be a digraph. The vertex set and the arc set of G are denoted by $V(G)$ and $A(G)$, respectively. Suppose that $e = (u,v) \in A(G)$. Then it is said that u is adjacent to v, v is adjacent from u, e is incident from u and e is incident to v. The set of vertices adjacent from (to) v is denoted by $N_G^+(v)$ $(N_G^-(v))$. The cardinality of $N_G^+(v)$ $(N_G^-(v))$ is called the out-degree (in-degree) of v and denoted by $deg_G^+(v)(deg_G^-(v))$. If $deg_G^+(v) = deg_G^-(v) = r$ for any vertex v of G, then G is called an r-regular digraph. A vertex whose out-degree is 0 is called a sink. Also, a vertex whose in-degree is 0 is called a source. The set of arcs incident from v is denoted by $A_G^+(v)$. Also $A_G^-(v)$ denotes the set of arcs incident to v. Let u, v be vertices of G. The distance from u to v denoted by $d_G(u,v)$ is the length of a shortest path from u to v. (If there is no path from u to v, then $d_G(u,v) = \infty$.) Let $S \subseteq V(G)$. Then the distance from S to v is defined by $\min_{u \in S} d_G(u,v)$. Let $V \subseteq V(G)$ and $A \subseteq A(G)$. Then $\langle V \rangle$ and $\langle A \rangle$ stand for the subdigraph induced by V and the subdigraph arc-induced by A, respectively. The underlying graph of G is a graph obtained from G by replacing each arc to a corresponding edge, deleting loops, and replacing multi-edges to single edges. If the underlying of a digraph is connected, then the digraph is called a weakly connected digraph. Let $F_1, F_2, \ldots, F_m$ be subdigraphs of G such that $\cup_{1 \le i \le m} A(F_i) = A(G)$,

$A(F_i) \cap A(F_j) = \emptyset$ for $1 \le i < j \le m$. Then it is said that G can be decomposed into $F_1, F_2, \ldots, F_m$. Also we say that G has a decomposition $[F_1, F_2, \ldots, F_m]$. If $F_i \cong F$, $1 \le i \le m$, then the decomposition of G is called an isomorphic decomposition of G into F and we say that G has an F-decomposition.

The line digraph of G denoted by $L(G)$ is a digraph whose vertex set is $A(G)$ such that vertex (u, v) is adjacent to vertex (x, y) iff $v = x$. The k-iterated line digraph of G denoted by $L^k(G)$ is recursively defined as $L^k(G) = L(L^{k-1}(G))$. Note that a vertex of $L^k(G)$ is corresponding to a walk of length k in G. Let W_v be the corresponding walk of G for vertex v of $L^k(G)$. Then vertex u is adjacent to vertex v in $L^k(G)$ iff the last subwalk of length $k-1$ of W_u is equal to the first subwalk of length $k-1$ of W_v. A vertex of $L^k(G)$ which is corresponding to a walk in a cycle of G is called a cyclical vertex of $L^k(G)$. Let v be a cyclical vertex of $L^k(G)$. Then $c(v)$ denotes the length of the cycle which contains W_v. Let P be a statement that can be true or false. Then $[P]$ stands for 1 if P is true, 0 otherwise.

Let K_p^* denote a complete symmetric digraph of order p. A digraph of order 1 and size d is denoted by K_1^d. That is, K_1^d has one vertex and d loops. Then, using the line digraph operation, the de Bruijn digraph $B(d, D)$ and the Kautz digraph $K(d, D)$ are defined as follows.

$$\begin{cases} B(d,D) = L^D(K_1^d), \\ K(d,D) = L^{D-1}(K_{d+1}^*). \end{cases}$$

From the point of walks, definitions by an alphabet of these digraphs are obtained. That is, The vertex set of $B(d, D)$ is $\{(v_1, v_2, \ldots, v_D) \mid v_i \in \{0, 1, \ldots, d-1\}, 1 \le i \le D\}$. Then there is an arc from vertex $(v_1, v_2, \ldots, v_D)$ to d vertices $(v_2, \ldots, v_D, \alpha)$ where $\alpha \in \{0, 1, \ldots, d-1\}$. Similarly, the vertex set of $K(d, D)$ is $\{(w_1, w_2, \ldots, w_D) \mid w_i \in \{0, 1, \ldots, d\}, 1 \le i \le D,\ w_j \ne w_{j+1}, 1 \le j < D\}$ such that there is an arc from vertex $(w_1, w_2, \ldots, w_D)$ to d vertices $(w_2, \ldots, w_D, \alpha)$ where $\alpha \in \{0, 1, \ldots, d\}$ and $\alpha \ne w_D$.

It is well known that the de Bruijn digraph can be decomposed into isomorphic spanning trees ([2]). We generalize this result from the point of the line digraph iteration. Then we present several isomorphic decompositions of the Kautz digraph applying isomorphic decompositions of the complete symmetric digraph.

Construction of arc-disjoint spanning trees rooted at the same vertex is related to the broadcasting scheme from a vertex of the root. In particular, we want to construct many spanning trees of maximum height as small as possible to get an efficient broadcasting scheme. For the de Bruijn digraph, Bermond and Fraigniaud [2] construct $d-1$ arc-disjoint spanning trees rooted at a given vertex of small height in $B(d, D)$. In particular, arc-disjoint spanning trees rooted at a vertex with a loop are directly given and the maximum height is optimal, that is $D+1$. For a vertex without a loop, an algorithm which construct arc-disjoint spanning trees are given. In a worst case, the maximum height is at most $D + 2\lfloor \frac{D}{2} \rfloor + 1$. In this paper, we construct arc-disjoint spanning trees rooted at a cyclical vertex of small height in the Kautz digraph. Let v be a cyclical vertex of $K(d, D)$. Then we construct d arc-disjoint spanning trees rooted at v of height at most $D + c(v) + [c(v) = 2] - [c(v) = d+1]$ in $K(d, D)$.

In section 2, we show that a decomposition of G induces a decomposition of $L(G)$. Also we define a class of digraphs named fountains and show some properties on the fountains and the line digraph operation. Then we present several isomorphic fountain-decompositions of the Kautz digraph applying isomorphic fountain-decompositions of the complete symmetric digraph. In section 3, applying decomposable results shown in section 2, we construct arc-disjoint spanning trees rooted at a cyclical vertex of small height in the Kautz digraph.

2 Isomorphic decompositions of Kautz digraphs

We introduce the following variation of the line digraph operation.

Definition 2.1 *Let H be a digraph. Let G be a subdigraph of H. Then $L_H(G)$ is defined as follows:*

$$\begin{cases} V(L_H(G)) = \{(u,v) \mid (u,v) \in A(H),\ u \in V(G)\}, \\ A(L_H(G)) = \{((u,v),(v,w)) \mid (u,v) \in A(G),\ (v,w) \in A(H)\}. \end{cases}$$

A decomposition of $L(G)$ is obtained from a decomposition of G.

Lemma 2.2 *Let H be a digraph. If H can be decomposed into $G_1, G_2, \ldots, G_m$, then $L(H)$ can be decomposed into $L_H(G_1), L_H(G_2), \ldots, L_H(G_m)$.*

Proof. Now $A(L(H)) = \{((u,v),(v,w)) \mid (u,v),(v,w) \in A(H)\}$. Then we can divide $A(L(H))$ according to the first element (u,v). Let $A_{(u,v)} = \{((u,v),(v,w)) \mid (v,w) \in A(H)\}$. Then

$$A(L(H)) = \cup_{e \in A(H)} A_e, \quad A_e \cap A_f = \emptyset \text{ if } e \neq f.$$

Since $A(H)$ can be divided into $A(G_1), A(G_2), \ldots, A(G_m)$, $A(L(H))$ can be divided into

$$\cup_{e \in A(G_1)} A_e,\ \cup_{e \in A(G_2)} A_e, \ldots, \cup_{e \in A(G_m)} A_e.$$

Here $L_H(G_i) = \langle \cup_{e \in A(G_i)} A_e \rangle$. Therefore $L(H)$ can be decomposed into $L_H(G_1), L_H(G_2), \ldots, L_H(G_m)$.
□

Definition 2.3 *Let G be a digraph. Then $\sigma_r(G)$ is a digraph obtained from G by adding, for each vertex v, new $\max(0, r - deg_G^+(v))$ arcs (with new distinct vertices) incident from v.*

If H is r-regular, then $L_H(G)$ is represented $L(\sigma_r(G))$. Also if H is r-regular, then $L(H)$ is r-regular too. Thus the following corollary holds.

Corollary 2.4 *Let H be an r-regular digraph. If H has a decomposition $[G_1, G_2, \ldots, G_m]$, then $L^k(H)$ has a decomposition $[(L \cdot \sigma_r)^k(G_1), (L \cdot \sigma_r)^k(G_2), \ldots, (L \cdot \sigma_r)^k(G_m)]$.*

Here we introduce a class of digraphs called fountains.

Definition 2.5 *Let G be a nontrivial weakly connected digraph. If $deg_G^-(v) = 1$ for any vertex v except for sinks, then G is called a fountain. If $deg_G^-(v) = 1$ for any vertex v, then G is called a proper fountain.*

The next proposition is well known.

Proposition 2.6 (Harary and Norman)
Let G be a nontrivial weakly connected digraph. Then $G \cong L(G)$ if and only if $deg_G^-(v) = 1$ for any vertex v or $deg_G^+(v) = 1$ for any vertex v.

Proper fountains are invariant with respect to the line digraph operation. As shown in the next lemma, fountains are invariant with respect to the operation $L \cdot \sigma_r$ such that "proper" or "non-proper" is also invariant. Let $\vec{K}(m,n)$ denote a digraph obtained from a complete bipartite graph with partite sets of order m and n by replacing edges with arcs such that all vertices in a partite set of order m are sources.

Lemma 2.7 *Let F be a fountain. Then $L(\sigma_r(F))$ is obtained from F by doing the following operations.*

1. *Apply σ_r-operation to F except for end-vertices.*

2. *For each end-vertex w, replace w with $\vec{K}(deg_F^-(w), r)$ and let each arc incident to w be incident to each source of $\vec{K}(deg_F^-(w), r)$ where distinct arcs are incident to distinct sources.*

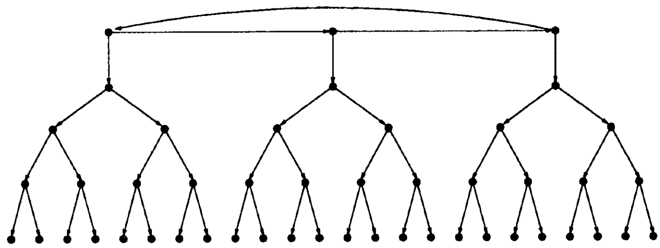

Figure 1: $F_3(2,4)$

Proof. Let H denote $\sigma_r(F) - End(F)$. Since $F - End(F)$ is a proper fountain, H is also a proper fountain. We consider a decomposition of $\sigma_r(F)$ into H and $\cup_{x \in End(F)} \langle A^-_{\sigma_r(F)}(x) \cup A^+_{\sigma_r(F)}(x) \rangle$. By Lemma 2.2, $L(\sigma_r(F))$ is decomposed into $L_{\sigma_r(F)}(H), \cup_{x \in End(F)} L_{\sigma_r(F)}(\langle A^-_{\sigma_r(F)}(x) \cup A^+_{\sigma_r(F)}(x) \rangle)$. Now

$$\begin{cases} V(L_{\sigma_r(F)}(H)) = V(L(H)) \cup \{(v,w) \in A(F) \mid w \in End(F)\}, \\ A(L_{\sigma_r(F)}(H)) = A(L(H)) \cup \{((u,v),(v,w)) \mid w \in End(F)\}. \end{cases}$$

Here $L(H) \cong H$.

Also

$$L_{\sigma_r(F)}(\langle A^-_{\sigma_r(F)}(x) \cup A^+_{\sigma_r(F)}(x) \rangle) \cong \vec{K}(deg^-_F(x), r).$$

And the set of sources of $L_{\sigma_r(F)}(\langle A^-_{\sigma_r(F)}(x) \cup A^+_{\sigma_r(F)}(x) \rangle)$ is $\{(v,x) \mid (v,x) \in A(F)\}$. Therefore the proposition holds. □

A complete m-ary out-tree is obtained from a complete m-ary tree by replacing edges to arcs such that there is a directed path from the root to any other vertex. Let $F_1(m,k)$ be a digraph obtained from a complete m-ary out-tree of height k by adding a loop to the root and deleting one complete m-ary out-tree of height $(k-1)$. Also let $F_p(m,k)$ be a digraph obtained from p $F_1(m,k)$ by deleting each loop of $F_1(m,k)$ and adding p arcs such that the set of the roots induces a cycle of order p.

The de Bruijn digraph $B(d,D)$ is a D-iterated line digraph of K_1^d. Also $(L \cdot \sigma_d)^D(K_1^1) \cong F_1(d,D)$ Thus the following proposition holds.

Proposition 2.8 (Bermond and Fraigniaud [2])

$B(d,D)$ has an $F_1(d,D)$-decomposition.

Similarly, fountain-decompositions of a Kautz digraph can be induced from fountain-decompositions of a complete symmetric digraph. Clearly, a complete symmetric digraph has C_2-decomposition. Some cycle-decompositions of a complete symmetric digraph have been known.

Theorem 2.9 (Bermond [1], Tillson [7])

- *K_p^* has a C_3-decomposition if $p \equiv 0$ or $1 \pmod 3$ and $p \neq 6$* [1].
- *K_p^* has a C_p-decomposition if $p \neq 4$ or 6* [7].

From these cycle-decompositions, we can get isomorphic decompositions of Kautz digraphs.

Theorem 2.10

- *$K(d,D)$ has an $F_2(d,D-1)$-decomposition.*

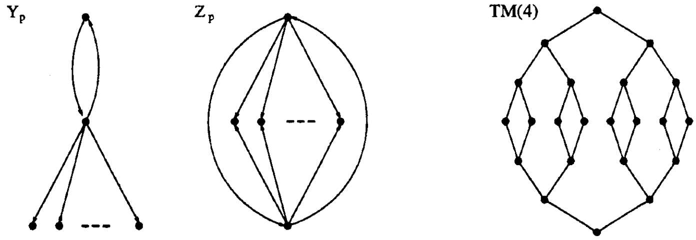

Figure 2: Y_p, Z_p and $TM(4)$

- *$K(d, D)$ has an $F_3(d, D-1)$-decomposition if $d \equiv 0$ or $2 \pmod 3$ and $d \neq 5$.*
- *$K(d, D)$ has an $F_{d+1}(d, D-1)$-decomposition if $d \neq 3$ or 5.*

Let Y_p and Z_p be fountains shown in Figure 2. Then it is easily checked that K_p^* has isomorphic decompositions into Y_p if $p \geq 3$ and Z_p if p is even and $p \geq 4$. Thus we can also get isomorphic decompositions of $K(d, D)$ applying these decompositions of K_{d+1}^*.

The back-to-back complete m-ary out-tree $BTB(m, D)$ is a digraph obtained from two complete m-ary out-tree of height D by identifying their corresponding leaves. Let T^* be a fountain with a sink of in-degree greater than one. Then $(L \cdot \sigma_m)^k(T^*))$ contains a digraph isomorphic to $BTB(m, k)$.

The underlying graph of $BTB(2, D)$ is called a tree machine $TM(D)$. The tree machine is well-known as an interconnection network. It is easily checked that $B(2,2)$ and $K(2,1)$ contain a fountain with a sink of in-degree greater than one. Thus, from Corollary 2.4 and Lemma 2.7, we can get results on the embeddability of a tree machine in the binary de Bruijn and Kautz graphs. (The de Bruijn and Kautz graphs are the underlying graphs of de Bruijn and Kautz digraphs, and denoted by $UB(d, D)$ and $UK(d, D)$, respectively.) For the binary de Bruijn digraph, the embeddability is shown in [6].

Corollary 2.11 (Samatham and Pradhan [6])
$TM(D-2) \subseteq UB(2, D)$.

Corollary 2.12 $TM(D-1) \subseteq UK(2, D)$.

3 Arc-disjoint spanning trees of Kautz digraphs

A decomposition of a digraph G into spanning proper fountains is abbreviated by an SPF-decomposition of G. Let F be a proper fountain. The cycle of F is denoted by $C(F)$. (Note that A proper fountain is unicyclic.) The length of $C(F)$ is denoted by $c(F)$. A digraph $F - A(C(F))$ is a digraph obtained from F by deleting all elements of $A(C(F))$. Thus $F - A(C(F))$ is a union of out-trees rooted at a vertex of $C(F)$. The maximum height of the out-trees of $F - A(C(F))$ is denoted by $h(F)$. Let w be a vertex of $C(F)$. Then the out-tree rooted at w in $F - A(C(F))$ is denoted by $sT_F(w)$. A digraph $F - V(C(F))$ is a digraph obtained from F by deleting all elements of $V(C(F))$. Thus $F - V(C(F))$ is also a union of out-trees. An out-tree rooted at x is denoted by $T_F(x)$. Let u be a vertex of $L^k(G)$. Then $l(W_u)$ denotes the last vertex of the walk W_u.

Lemma 3.1 *Let G be a loopless digraph. Suppose that G has an SPF-decomposition $[F_1, F_2, \ldots, F_m]$ such that all cycles of the fountains have a common vertex v. Let w be a cyclical vertex of $L^k(G)$ such that $W_w \subseteq C(F_1)$ and $l(W_w) = v$. Let $N_{F_1}^-(v) = \{v'\}$. Then there exist m arc-disjoint spanning trees rooted at w of height at most $k + \max_{1 \leq i \leq m}(c(F_i) + h(F_i) + [i = 1][deg_{F_1}^+(v') \neq 1])$ in $L^k(G)$.*

Proof. Let $[F_1, F_2, \ldots, F_m]$ be an SPF-decomposition of G such that all cycles of the fountains contain a vertex v. Also, let $[H_1, H_2, \ldots, H_m]$ be the SPF-decomposition of $L^k(G)$ induced by $[F_1, F_2, \ldots, F_m]$ where H_i is induced by F_i for $i = 1, 2, \ldots, m$.

Let $N_G^+(v) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ such that $(v, \alpha_i) \in A(F_i)$ for $i = 1, 2, \ldots, m$. Let w be a vertex such that $W_w \subseteq C(F_1)$ and $l(W_w) = v$. Also let $N_{F_1}^-(v) = \{v'\}$ and $N_{H_1}^-(w) = \{w'\}$. Let $N_{L^k(G)}^+(w) = \{w_1, w_2, \ldots, w_m\}$ such that $l(W_{w_i}) = \alpha_i$ for $i = 1, 2, \ldots, m$.

Let $B_i = \langle\{(w, w_i)\} \cup A(T_{H_1}(w_i))\rangle$ for $i = 2, \ldots, m$. Then we define T_i, $i = 1, 2, \ldots, m$ as follows and show that these trees are arc-disjoint spanning trees rooted at w of $L^k(G)$.

$$\begin{cases} T_i = \langle A(B_i) \cup (A(H_i) \setminus \{(x, y) \mid y \in V(B_i)\})\rangle, \\ T_1 = \langle (A(H_1) \setminus ((\cup_{2 \le i \le m} A(B_i)) \cup \{(w', w)\})) \cup (\cup_{2 \le i \le m}(A(H_i) \cap \{(x, y) \mid y \in V(B_i) \setminus \{w\}\})\rangle. \end{cases}$$

Since $A(B_i) \cap A(B_j) = \emptyset$ and $A(H_i) \cap A(H_j) = \emptyset$ for $2 \le i < j \le m$, $T_2, \ldots, T_m$ are arc-disjoint. Also, it is easily checked that T_1 and T_i are arc-disjoint for $2 \le i \le m$.

First, we show that T_i, $i \ne 1$ is a spanning tree rooted at w of height at most $k + c(F_i) + h(F_i)$. Let $Y(T_{H_1}(w_i))$ denote the set of leaves of $T_{H_1}(w_i)$. Then $Y(T_{H_1}(w_i))$ is corresponding to the set of walks of G whose first and second vertices are v and α_i, respectively. Therefore, there is a cyclical vertex x_i in $Y(T_{H_1}(w_i))$ such that $W_{x_i} \subseteq C(F_i)$. Since H_i is a spanning proper fountain of $L^k(G)$, for any vertex of $L^k(G)$, there is a path from x_i in H_i. Thus T_i is a spanning tree rooted at w of $L^k(G)$.

In H_1, the distance from w to any vertex of $T_{H_1}(w_i)$ is at most k. In particular, the distance from w to x_i is k. Now consider the locations of leaves of $T_{H_1}(w_i)$ in H_i. Let $\gamma_0\gamma_1 \cdots \gamma_{l-1}$ be the cycle of H_i such that $\gamma_0 = x_i$. Let $V_j(sT_{H_i}(\gamma_r))$ denote the set of vertices of $sT_{H_i}(\gamma_r)$ whose distances from γ_r are j. Let $W_{\gamma_0} = g_0g_1 \cdots g_k$. Note that $g_0 = v$ and $g_1 = \alpha_i$. Then $\gamma_{-j(mod\ l)}$ is corresponding to a walk obtained from $g_0g_1 \cdots g_k$ by j-times right-shifting. Thus $W_{\gamma_{-j(mod\ l)}}$ is represented as follows.

$$h_jh_{j-1} \cdots h_1g_0g_1 \cdots g_{k-j}.$$

If there is a path of length p from vertex u to vertex u' in $L^k(G)$, then $W_{u'}$ is obtained from W_u by p-times left-shifting. Thus any element of $V_j(sT_{H_i}(\gamma_{-j(mod\ l)}))$ is a leaf of $T_{H_1}(w_i)$ if $j < k$. Let y be a leaf of $T_{H_1}(w_i)$ such that $y \ne x_i$ and $W_y = y_0y_1 \cdots y_k$. Note that $y_0 = v$ and $y_1 = \alpha_i$. Let r be an integer such that $y_i = g_i$ for $i \le r$ and $y_{r+1} \ne g_{r+1}$. Then $y \in V_{k-r}(sT_{H_i}(\gamma_{r-k(mod\ l)}))$. Therefore

$$Y(T_{H_1}(w_i)) = \cup_{0 \le j < k} V_j(sT_{H_i}(\gamma_{-j(mod\ l)})).$$

Thus, for any vertex of $\cup_{0 \le p < l,\ 0 \le j \le k} V_j(sT_{H_i}(\gamma_p))$, the distance from $Y(T_{H_1}(w_i))$ is at most l, that is at most $c(F_i)$. The height of $sT_{H_i}(\gamma_j)$ is at most $k + h(F_i)$. Therefore the distance from $Y(T_{H_1}(w_i))$ to any leaf of H_i is at most $c(F_i) + h(F_i)$. Hence the height of T_i, $i \ne 1$, is at most $k + c(F_i) + h(F_i)$.

Next we show that T_1 is a spanning tree rooted at w of height at most $k+c(F_1)+h(F_1)+[deg_{F_1}^+(v') \ne 1]$. Since H_1 is a spanning proper fountain of $L^k(G)$, there is a path from w to any vertex of $V(L^k(G)) \setminus (\cup_{2 \le i \le m} V(T_{H_1}(w_i)))$ in T_1. Here the length is at most $c(F_1) - 1 + k + h(F_1)$. Also, H_i, $i \ne 1$ is a spanning proper fountain. Thus there is an arc incident to any vertex of $V(T_{H_1}(w_i))$ in $A(H_i) \cap \{(x, y) \mid y \in V(B_i),\ y \ne w\}$. Let z be a vertex of $Y(T_{H_1}(w_i))$ where $i \ne 1$. Then the first vertex and the second vertex of W_z are v and α_i, respectively. Since $\alpha_i \ne v$, there is no arc between vertices of $\cup_{2 \le i \le m} Y(T_{H_1}(w_i))$. Also, any vertex of $V(T_{H_1}(w_i)) \setminus Y(T_{H_1}(w_i))$ is a leaf of $H_i, i \ne 1$. Thus an arc in $A(H_i) \cap \{(x, y) \mid y \in V(B_i), y \ne w\}$ which is incident to z is incident from a vertex of $V(H_1) \setminus (\cup_{2 \le i \le m} V(T_{H_1}(w_i)))$. Therefore, for any vertex of $\cup_{2 \le i \le m} Y(T_{H_1}(w_i))$, there is a path from w. Here the length is at most $c(F_1) + k + h(F_1)$. Since any vertex z' of $V(T_{H_1}(w_i)) \setminus Y(T_{H_1}(w_i))$ is a leaf of H_i, $i \ne 1$, the arc incident to z' in $A(H_i) \cap \{(x, y) \mid y \in V(B_i),\ y \ne w\}$ is incident from a vertex of $V(H_1) \setminus (\cup_{2 \le i \le m}(V(T_{H_1}(w_i)) \setminus Y(T_{H_1}(w_i))))$. Thus there is a path from w to z' in T_1 of length at most $c(F_1) + k + h(F_1) + 1$.

Suppose that $N_G^+(v') = \{\beta_1, \beta_2, \ldots, \beta_{m'}, v\}$. Let w_i' be a vertex of $N^+_{L^k(G)}(w')$ such that $l(W_{w_i'}) = \beta_i$. Let $h(sT_{F_1}(v')) = 0$. Then $h(sT_{H_1}(w')) = k$ and $h(T_{H_1}(w_i')) = k-1$. Now the first vertex and the second vertex of a walk corresponding to a leaf of $T_{H_1}(w_i')$ are v' and β_i. Therefore, there is no arc incident from a leaf of $T_{H_1}(w_i')$ to a leaf of $T_{H_1}(w_j)$. Thus the length of a path from w to any leaf of $T_{H_1}(w_j)$ is at most $c(F_1) - 1 + k + h(F_1)$. Hence, in this case, the height of T_1 is at most $c(F_1) + k + h(F_1)$. □

Corollary 3.2 *Let G be a loopless digraph of order p. Suppose that G has a hamiltonian cycle decomposition $[H_1, H_2, \ldots, H_m]$. Let w be a cyclical vertex of $L^k(G)$ which is corresponding to a walk in an H_i. Then there exist m arc-disjoint spanning trees rooted at w of height at most $k+p$ in $L^k(G)$.*

In Lemma 3.1, we assume that G is loopless. If G has a loop and there exists an SPF-decomposition which satisfy the condition in Lemma 3.1, then G must be isomorphic to K_1^d for some positive integer d. In this case, we can also construct arc-disjoint spanning trees T_i except T_1 according to the definitions in the proof of Lemma 3.1. (Note that a walk of K_1^d is represented as a sequence of arcs in this case.) These arc-disjoint spanning trees are the same trees shown in [2].

Proposition 3.3 (Bermond and Fraigniaud [2])

There exist $d-1$ arc-disjoint spanning trees of height $D+1$ rooted at a vertex with a loop in $B(d,D)$.

Constructing SPF-decompositions of K^*_{d+1} which satisfy the condition in Lemma 3.1, we can get arc-disjoint spanning trees of $K(d,D)$.

Theorem 3.4 *Let v be a cyclical vertex of $K(d,D)$. Then there exist d arc-disjoint spanning trees rooted at v of height at most $D + c(v) + [c(v) = 2] - [c(v) = d+1]$ in $K(d,D)$.*

Proof. We present SPF-decompositions of K^*_{d+1} which satisfy the condition in Lemma 3.1.

Since any permutation on $V(K^*_{d+1})$ is an automorphism of K^*_{d+1}, it is sufficient to show that for any $l \in \{2, \ldots, d+1\}$, there is an SPF-decomposition of K^*_{d+1}, $[F_1, F_2, \ldots, F_m]$ such that all cycles of the fountains contain a common vertex and $c(F_1) = l$.

Case 1: $l \neq 4$ or 6. If $l = d+1$, then we can employ a hamiltonian cycle decomposition of K^*_{d+1}. In this case, the maximum height of d arc-disjoint spanning trees is at most $(D-1) + (d+1) = D + d$.

Suppose that $2 \leq l \leq d$. Let $V(K^*_{d+1}) = \{\alpha_0, \alpha_1, \ldots, \alpha_{l-1}, \beta_0, \beta_1, \ldots, \beta_{m-1}\}$, where $m = d+1-l$. Let $[H_1, H_2, \ldots, H_{l-1}]$ be a hamiltonian cycle decomposition of $\langle\{\alpha_0, \alpha_1, \ldots, \alpha_{l-1}\}\rangle_{K^*_{d+1}}$. Without loss of generality, we can let $(\alpha_0, \alpha_1) \in A(H_1)$. We define H_i' as follows.

$$\begin{cases} V(H_i') = V(K^*_{d+1}), \\ A(H_i') = A(H_i) \cup \{(\alpha_i, \beta_j) \mid 0 \leq j < m\}. \end{cases}$$

Also we define F_j, $0 \leq j < m$, as follows.

$$\begin{cases} V(F_j) = V(K^*_{d+1}), \\ A(F_j) = \{(\alpha_0, \beta_j)\} \cup \{(\beta_j, v) \mid v \in V(K^*_{d+1}) \setminus \{\beta_j\}\}. \end{cases}$$

Then H_i', $1 \leq i < l$ and F_j, $0 \leq j < m$ are arc-disjoint spanning proper fountains such that all cycles of H_i' and F_j have α_0 as a common vertex. That is, $[H_1', H_2', \ldots, H_{l-1}', F_0, F_1, \ldots, F_{m-1}]$ is an SPF-decomposition of K^*_{d+1} which satisfy the condition in Lemma 3.1. Therefore, there exist d arc-disjoint spanning trees rooted at any vertex which is corresponding to a walk in a cycle of length l of K^*_{d+1}. Here the maximum height of the trees is at most $(D-1) + \max(l+1, 2+1) = D + l$ if $l \neq 2$. If $l = 2$, then the maximum height is at most $(D-1) + \max(4,3) = D+3$.

Case 2: $l = 4$. We define H_i, $i = 1,2,3$, as follows.

$$\begin{cases} V(H_i) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}, \ i = 1,2,3, \\ A(H_1) = \{(\alpha_0, \alpha_1), (\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_3, \alpha_0)\}, \\ A(H_2) = \{(\alpha_0, \alpha_2), (\alpha_2, \alpha_1), (\alpha_1, \alpha_0), (\alpha_1, \alpha_3)\}, \\ A(H_3) = \{(\alpha_0, \alpha_3), (\alpha_3, \alpha_2), (\alpha_2, \alpha_0), (\alpha_3, \alpha_1)\}. \end{cases}$$

Then $[H_1, H_2, H_3]$ is a SPF-decomposition of K_4^* such that all cycles of H_1, H_2 and H_3 contain α_0. If $d = 3$, then we employ this SPF-decomposition. Then, the maximum height of the arc-disjoint spanning trees is at most $(D-1)+4 = D+3$.

Suppose that $d > 3$. We define H_i', $i = 1, 2, 3$ and F_j, $j = 0, 1, \ldots, d-4$, similar to the previous case. Then $[H_1', H_2', H_3', F_0, \ldots, F_{d-4}]$ is an SPF-decomposition of K_{d+1}^* which satisfy the condition in Lemma 3.1. Therefore, there exist d arc-disjoint spanning trees rooted at any vertex which is corresponding to a walk in a cycle of length 4. Here the maximum height of the trees is at most $(D-1) + \max(5,4,4,3) = D+4$.

Case 3: $l = 6$. We define H_i, $i = 1, 2, \ldots, 5$, as follows.

$$\begin{cases} V(H_i) = \{\alpha_j \mid 0 \le j < 6\},\ i = 1, 2, \ldots, 5, \\ A(H_1) = \{(\alpha_0,\alpha_1),(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_5,\alpha_0)\}, \\ A(H_2) = \{(\alpha_0,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_0),(\alpha_2,\alpha_5),(\alpha_4,\alpha_1),(\alpha_1,\alpha_3)\}, \\ A(H_3) = \{(\alpha_0,\alpha_3),(\alpha_3,\alpha_0),(\alpha_3,\alpha_5),(\alpha_5,\alpha_2),(\alpha_5,\alpha_1),(\alpha_1,\alpha_4)\}, \\ A(H_4) = \{(\alpha_0,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_1),(\alpha_1,\alpha_0),(\alpha_1,\alpha_5),(\alpha_4,\alpha_2)\}, \\ A(H_5) = \{(\alpha_0,\alpha_5),(\alpha_5,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_0),(\alpha_5,\alpha_4),(\alpha_2,\alpha_1)\} \end{cases}$$

Then $[H_1, H_2, \ldots, H_5]$ is an SPF-decomposition of K_6^* such that all cycles of the fountains contain α_0.

Similarly to the case 2, we can get d arc-disjoint spanning trees of height at most $D+5$ if $d = 5$, $D+6$ if $d > 5$ using this SPF-decomposition of K_6^*. □

4 Conclusion

In this paper, we have shown several isomorphic decompositions of the Kautz digraph. Also we have given d arc-disjoint spanning trees rooted at a cyclical vertex of small height in $K(d, D)$. It remains to construct arc-disjoint spanning trees rooted at a non-cyclical vertex of small height in the Kautz digraph.

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