

## 再構成メッシュ上で $O((\log \log n)^2)$ 時間で凸包を求めるアルゴリズム

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本論文では、プロセッサ台数  $\frac{\sqrt{n}}{\log \log n} \times \frac{\sqrt{n}}{\log \log n}$  の再構成メッシュ上で  $O((\log \log n)^2)$  時間で凸包を求めるアルゴリズムを示す。さらに、 $n$  プロセッサの再構成メッシュが凸包を求めるのに少なくとも  $\Omega(\log \log n)$  時間必要であることをしめす。

## An $O((\log \log n)^2)$ Time Convex Hull Algorithm on Reconfigurable Meshes

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We propose an almost optimal algorithm that computes the convex hull in  $O((\log \log n)^2)$  time on a reconfigurable mesh of size  $\frac{\sqrt{n}}{\log \log n} \times \frac{\sqrt{n}}{\log \log n}$ . We also show that any algorithm that computes the convex hull of a set of  $n$  sorted points on an  $n$ -processor reconfigurable mesh must take  $\Omega(\log \log n)$  time.

### 1 Introduction

In essence, a reconfigurable mesh (RM) consists of a mesh augmented by the addition of a dynamic bus system whose configuration changes in response to computational and communication needs. More precisely, a RM of size  $n \times m$  consists of  $nm$  identical SIMD processors positioned on a rectangular array with  $n$  rows and  $m$  columns. As usual, it is assumed that every processor knows its own coordinates within the mesh: we let  $PE(i, j)$  denote the processor in row  $i$  and column  $j$ , with  $PE(1, 1)$  in the north-west corner of the mesh.

Each processor  $PE(i, j)$  is connected to its four neighbors  $P(i-1, j)$ ,  $P(i+1, j)$ ,  $P(i, j-1)$ , and  $P(i, j+1)$ , provided they exist, and has 4 ports denoted by N, S, E, and W in Figure 1. Local connections between these ports can be established, under program control, creating a powerful bus system that changes dynamically to accommodate various computational needs. For practical reasons, only one processor is allowed to broadcast on a given bus at any one time. All the results in this paper assume a model that allows at most two pairs of connections to be set in each processor at any one time.

Given its importance, and its far-reaching applications, the convex hull problem has been studied extensively in the literature, both sequentially and in parallel [2, 3, 8]. On the reconfigurable mesh, the convex hull problem has been addressed in two different contexts: for *sparse input* and for *dense input*. While the

sparse case allows one to use more processors than input points, in the dense case the number of processors and the number of input points are, essentially, the same. For sparse input, Olariu *et al.* [6] and Jang *et al.* [3] proposed  $O(1)$  time algorithms to compute the convex hull of a set of  $\sqrt{n}$  points on a reconfigurable mesh of size  $\sqrt{n} \times \sqrt{n}$ . Nakano [5] showed that, if the  $\sqrt{n}$  points are sorted beforehand, then, for every fixed  $\epsilon > 0$ , the convex hull can be computed in  $O(1)$  time on a reconfigurable mesh of size  $\sqrt{n} \times n^\epsilon$ .

In the dense case, Miller and Stout [4] proposed an  $O(\log^2 n)$  time algorithm computing the convex hull of a sorted set of  $n$  points, pretiled in proximity order on a reconfigurable mesh of size  $\sqrt{n} \times \sqrt{n}$ . Olariu *et al.* [7] using a different approach proposed algorithms featuring, essentially, the same  $O(\log^2 n)$  performance. Recently, Nakano [5] showed that the convex hull of a sorted set of  $\sqrt{mn}$  points can be computed in  $O(\frac{\log^2 n}{\log m} + \log^2 m)$  time on a reconfigurable mesh of size  $\sqrt{m} \times \sqrt{n}$ . In particular, for  $m = 2^{\log^{\frac{2}{3}} n}$  the computing time becomes  $O(\log^{\frac{2}{3}} n)$ , which is the fastest known for dense input.

For more than eight years it was open to obtain a convex hull algorithm running in sub-logarithmic time on dense input. Our main contribution is to provide the first breakthrough: we propose an almost optimal algorithm running in  $O((\log \log n)^2)$  time on a reconfigurable mesh of size  $\sqrt{n} \times \sqrt{n}$ . We then go on to show that with minor changes this algorithm can be

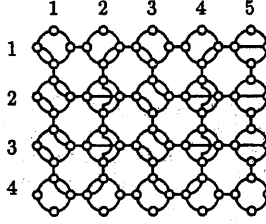


Figure 1: A reconfigurable mesh of size  $4 \times 5$  and several subbuses

implemented to run in  $O((\log \log n)^2)$  time on a reconfigurable mesh of size  $\frac{\sqrt{n}}{\log \log n} \times \frac{\sqrt{n}}{\log \log n}$ . Our second algorithm features a processor-time product of  $O(n)$ , being work-optimal.

We also show that our algorithms are close to time-optimality. Indeed, we prove that any algorithm that computes the convex hull of a set of  $n$  sorted points on an  $n$ -processor reconfigurable mesh must take  $\Omega(\log \log n)$  time. In fact, this lower bound holds even for the strongest reconfigurable meshes that allow port fusion.

## 2 The lower bound

In order to establish a time-lower bound for the problem of computing the upper hull of a set of  $n$  points on a reconfigurable mesh with  $n$  processors we rely on the following result of Valiant [9].

**Lemma 2.1** *The problem of finding the maximum of  $n$  items requires  $\Omega(\log \log n)$  time on the parallel comparison model, provided that the number of processors is at most  $n \log^{O(1)} n$ .*

We observe that, for comparison-based problems, the parallel comparison model with  $n$  processors can simulate any parallel machine with  $n$  processors including the reconfigurable mesh. Thus, Lemma 2.1 has the following consequence.

**Corollary 2.2** *The problem of finding the maximum of  $n$  items requires  $\Omega(\log \log n)$  time on the reconfigurable mesh, provided that the number of processors is at most  $n \log^{O(1)} n$ .*

We now reduce the problem of finding the maximum of  $\frac{n}{2}$  items to that of computing the upper hull of  $n$  sorted points in the plane. For this purpose, let  $A = \{a_1, a_2, \dots, a_{\frac{n}{2}}\}$  be an arbitrary input to the maximum problem. From this input we generate a set  $P$  of  $n$  sorted points in the plane by writing  $P = \{(i, a_i), (i + \frac{n}{2}, a_i) \mid 1 \leq i \leq \frac{n}{2}\}$ . Notice that the points  $(j, a_j)$  and  $(j + \frac{n}{2}, a_j)$  are both vertices of the upper hull of  $P$  if and only if  $a_j$  is the maximum of  $A$ . It

follows that any algorithm that computes the upper hull of  $P$  also computes the maximum of  $A$ . Thus, we have

**Theorem 2.3** *The problem of computing the upper hull of  $n$  sorted points in the plane requires  $\Omega(\log \log n)$  time on an  $n$ -processor reconfigurable mesh.*

## 3 Our geometric machinery

In this section we develop novel geometric results that will lay the foundation of our convex hull algorithms. Let  $P_1, P_2, \dots, P_m$  be disjoint upper hulls, each containing  $n$  points, such that for all  $i$ ,  $(1 \leq i \leq m-1)$ ,  $P_i$  is to the left of  $P_{i+1}$ . We are interested in computing the upper hull  $U(P)$  of  $P = P_1 \cup P_2 \cup \dots \cup P_m$ . For every  $i$ ,  $(1 \leq i \leq m)$ , enumerate the points of  $P_i$  in increasing  $x$ -coordinate as  $p(i, 1), p(i, 2), p(i, 3), \dots, p(i, n)$ . Observe that  $U(P)$  is a convex polygonal chain with left endpoint  $p(1, 1)$  and right endpoint  $p(m, n)$ . It is easy to see that the points belonging to  $U(P) \cap P_i$  are consecutive in  $U(P)$  and can be specified by the interval  $[g_i, h_i]$ : in other words,  $U(P) \cap P_i = \{p(i, g_i), p(i, g_i + 1), \dots, p(i, h_i)\}$ . The two points  $L_i = p(i, g_i)$  and  $R_i = p(i, h_i)$  are termed, respectively, the *left contact point* and the *right contact point* of  $P_i$  with respect to  $U(P)$ . Note that each  $P_i$  may have, with respect to  $U(P)$ , one, two, or no contact points. Clearly,  $P_i$  has exactly one contact point only if  $L_i = R_i$ .

The line segment  $p(i, h_i)p(j, g_j)$ ,  $(i < j)$ , is said to be an *upper hull tangent* of  $U(P)$  if for every  $k$ ,  $(i < k < j)$ ,  $U(P) \cap P_k$  is empty. In other words, all the points in  $P_{i+1} \cup P_{i+2} \cup \dots \cup P_{j-1}$  lie below the line segment  $p(i, h_i)p(j, g_j)$ .

Samples will play a crucial role in the sequel of this work. For our purpose, a *sample* is just a subset of a given set. In particular, let  $S(P_i)$  be a sample of  $P_i$  including the points  $p(i, 1)$  and  $p(i, n)$ . Let  $U(S(P))$  be the upper hull of  $S(P) = S(P_1) \cup S(P_2) \cup \dots \cup S(P_m)$ . It should be clear that  $U(S(P))$  is also a convex polygonal chain with left endpoint  $p(1, 1)$  and right endpoint  $p(m, n)$ . As noted,  $S(P_i)$  may have

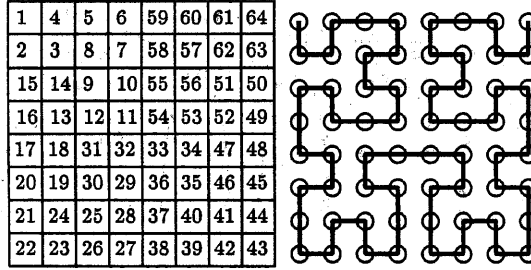


Figure 2: Proximity order and the corresponding bus embedding

one, two, or no contact points with respect to  $U(S(P))$ . If  $S(P_i)$  has no contact points with respect to  $U(S(P))$ , then it must be the case that all the points in  $S(P_i)$  lie below some upper hull tangent of  $U(S(P))$ . In this case, the closest point  $p(i, g'_i)$  to this upper hull tangent over all the points in  $S(P_i)$  is termed a *pseudo contact point* of  $S(P_i)$  with respect to  $U(S(P))$ . We refer the reader to Figure 3 for an illustration of these concepts. Here, the dark circles denote points in  $S(P)$ .

In order to specify some of our results we need the following definition. Let  $C_1$  and  $C_2$  be two convex polygonal chains with left and right endpoints  $p(1, 1)$  and  $p(m, n)$ . We say that  $C_2$  is an *upper envelope* for  $C_1$  if on every vertical line that intersect both  $C_1$  and  $C_2$  the intersection point with  $C_2$  is not below the intersection point with  $C_1$ . The following lemmas, in addition to shedding light on the relationship between  $U(S(P))$  and  $U(P)$ , will turn out to be key ingredients in our upper hull algorithms.

**Lemma 3.1** *For every choice of samples  $S(P_i)$ , ( $1 \leq i \leq m$ ),  $U(P)$  is an upper envelope of  $U(S(P))$ .*

**Proof.** If some vertical line intersects  $U(P)$  below  $U(S(P))$  then some vertex  $u$  of  $U(S(P))$  lies above some edge of  $U(P)$ , contradicting that  $U(P)$  is the convex hull of  $P_1 \cup P_2 \cup \dots \cup P_m$ .  $\square$

For a point  $p(i, f)$  of  $S(P_i)$  we let  $p(i, l(f))$  and  $p(i, r(f))$  denote, respectively, its left and right neighbor in  $S(P_i)$ . Let  $p(i, g'_i)$  and  $p(i, h'_i)$  be, respectively, the left and right contact points of  $S(P_i)$  with respect to  $U(S(P))$ . Clearly,  $U(S(P)) \cap S(P_i) = \{p(i, g'_i), p(i, r(g'_i)), p(i, r(r(g'_i))), \dots, p(i, l(h'_i)), p(i, h'_i)\}$ .

**Lemma 3.2** *Let  $p(i, g_i)$  and  $p(i, h_i)$  be the left and right contact points (if any) of  $P_i$  with respect to  $U(P)$ . If the points  $p(i, g'_i)$  and  $p(i, h'_i)$  are distinct then  $l(g'_i) < g_i \leq h_i < r(h'_i)$ . If  $p(i, g'_i)$  and  $p(i, h'_i)$  coincide or if  $S(P_i)$  has a*

*pseudo contact point  $p(i, g'_i)$  with respect to  $U(S(P))$ , then  $l(g'_i) < g_i \leq h_i < r(g'_i)$ .*

**Proof.** First, assume that the points  $p(i, g'_i)$  and  $p(i, h'_i)$  exist and are distinct and refer to Figure 4. Let  $p(i, h'_i)p(j, g'_j)$ , ( $i < j$ ), be an upper hull tangent of  $U(S(P))$ . Clearly, the points  $p(i, k)$  with  $r(h'_i) \leq k$  lie below the upper hull tangent  $p(i, h'_i)p(j, g'_j)$ . Hence, by Lemma 3.1,  $p(i, h_i)$  cannot be among them. Therefore,  $h_i < r(h'_i)$  must hold. A symmetric argument shows that  $l(g'_i) < g_i$ . Thus, we have  $l(g'_i) < g_i \leq h_i < r(h'_i)$ .

The case where the points  $p(i, g'_i)$  and  $p(i, h'_i)$  coincide is handled in a perfectly similar way and will not be repeated.

Finally, assume that  $S(P_i)$  has a pseudo contact point  $p(i, g'_i)$  and let  $\lambda$  be the upper hull tangent of  $U(S(P))$  confirming that  $p(i, g'_i)$  is a pseudo contact point. Clearly, the points  $p(i, k)$  with  $k \leq l(i, g'_i)$  or  $r(i, g'_i) \leq k$  must lie below  $\lambda$ . By Lemma 3.1, neither  $p(i, g_i)$  nor  $p(i, h_i)$  can be among them. It follows that  $l(g'_i) < g_i \leq h_i < r(g'_i)$ , as claimed.  $\square$

Let  $p(i, h_i)p(j, g_j)$ , ( $i < j$ ), be an upper hull tangent of  $U(P)$ . As we noted before,  $p(i, h_i)$  is the right contact point of  $P_i$ ,  $p(j, g_j)$  is the left contact point of  $P_j$ , and all the points in  $P_{i+1} \cup P_{i+2} \cup \dots \cup P_{j-1}$  lie below the line determined by  $p(i, h_i)$  and  $p(j, g_j)$ .

**Lemma 3.3** *Let  $p(i, h'_i)$  be the right contact point or the pseudo contact point of  $S(P_i)$  with respect to  $U(S(P))$ , and let  $p(j, g'_j)$  be the left contact point or the pseudo contact point of  $S(P_j)$  with respect to  $U(S(P))$ . At least one of the conditions  $l(h'_i) < h_i < r(h'_i)$  or  $l(g'_j) < g_j < r(g'_j)$  is satisfied.*

**Proof.** Let  $p(i, h_i)p(j, g_j)$ , ( $i < j$ ) be an upper hull tangent of  $U(P)$  and refer to Figure 5. By Lemma 3.2 we must have

$$h_i < r(h'_i) \text{ and } l(g'_j) < g_j. \quad (1)$$

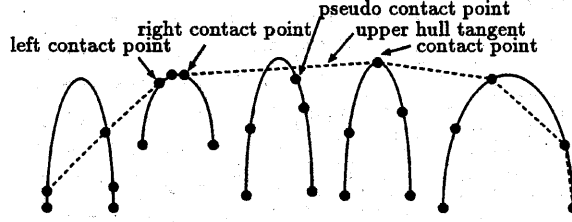


Figure 3: Illustrating contact points and upper hull tangents of  $U(S(P))$

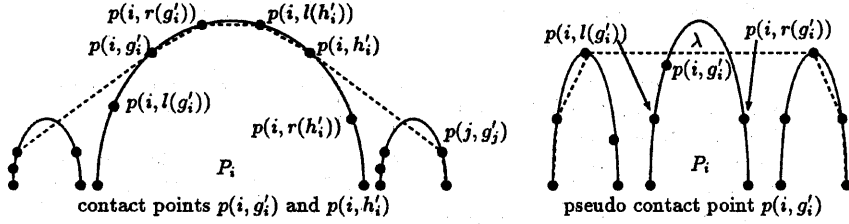


Figure 4: Illustrating the proof of Lemma 3.2

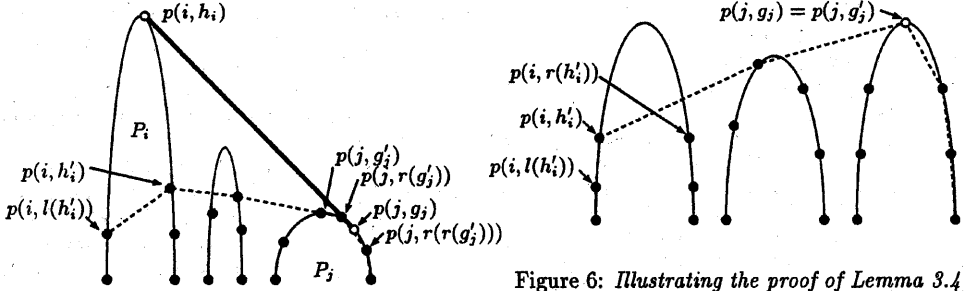


Figure 5: Illustrating the proof of Lemma 3.3

If both  $h_i \leq l(h'_i)$  and  $r(g'_j) \leq g_j$  are satisfied, then at least one of the points  $p(i, h_i)$  and  $p(j, g_j)$  must lie below the line segment determined by  $p(i, h'_i)$  and  $p(j, g'_j)$ , contradicting Lemma 3.1. Therefore, we must have

$$l(h'_i) < h_i \text{ or } g_j < r(g'_j). \quad (2)$$

Now, equations (1) and (2), combined, imply that  $l(h'_i) < h_i < r(h'_i)$  or  $l(g'_j) < g_j < r(g'_j)$  is satisfied, completing the proof of the lemma.  $\square$

Let  $p(i, h_i)p(j, g_j)$ , ( $i < j$ ), be an upper hull tangent of  $U(P)$  and assume that one of the points  $p(i, h_i)$  or  $p(j, g_j)$  is known.

**Lemma 3.4** Let  $p(i, h'_i)$  be the right contact point of  $S(P_i)$  and let  $p(j, g'_j)$  be the left contact point of  $S(P_j)$  with respect to  $U(S(P))$ . If  $p(j, g'_j)$  is also the right contact point of  $P_i$  with respect to

$U(P)$ , i.e.  $g'_j = g_j$ , then we must have  $l(h'_i) < h_i < r(h'_i)$ .

**Proof.** Referring to Figure 6, note that by Lemma 3.1 among the points in  $P_i$ , only those lying on or above the line  $p(i, h'_i)p(j, g_j)$  can be the right contact point  $p(i, h_i)$  of  $P_i$ .  $\square$

We shall say that  $S(P)$  is a *good sample* of  $P$  if all the contact points of  $P$  with respect to  $U(P)$  belong to  $S(P)$ . Clearly, in this case, the contact points of  $S(P)$  with respect to  $U(S(P))$  are precisely the contact points of  $P$  with respect to  $U(P)$ . Thus, once a good sample  $S(P)$  is available, the task of computing all the contact points of  $P$  with respect to  $U(P)$  and, therefore,  $U(P)$  itself, reduces to the task of computing  $U(S(P))$ .

## 4 The key idea: obtaining a good sample efficiently

We are now in a position to show how the geometric machinery developed in Section 3 can be exploited to obtain a good sample  $S(P)$  of  $P$ . For this purpose, we shall find it convenient to import the relevant notation and terminology developed in the context of Section 3.

We begin by selecting for every  $i$ , ( $1 \leq i \leq m$ ), a sample  $S(P_i)$  by retaining every  $n^{\frac{1}{2}}$ -th item in  $P_i$ . Note that the first and last point of  $P_i$  are also included in  $S(P_i)$ . Write  $S(P) = S(P_1) \cup S(P_2) \cup \dots \cup S(P_m)$ . Having computed the convex hull  $U(S(P))$  of  $S(P)$ , we determine for every  $S(P_i)$  its left and right contact points,  $p(i, g'_i)$  and  $p(i, h'_i)$  (if any) with respect to  $U(S(P))$ . Lemma 3.3 guarantees that the left and right contact points  $p(i, g_i)$  and  $p(i, h_i)$  of  $P_i$  with respect to  $U(P)$  satisfy  $l(g'_i) < g_i < r(g'_i)$  or  $l(h'_i) < h_i < r(h'_i)$ . This motivates us to refine the sample  $S(P_i)$  by adding more sample points from the sequence  $p(i, l(g'_i)), p(i, l(g'_i) + 1), \dots, p(i, g'_i), \dots, p(i, r(g'_i) - 1), p(i, r(g'_i))$  and, similarly, from the sequence  $p(i, l(h'_i)), p(i, l(h'_i) + 1), \dots, p(i, h'_i), \dots, p(i, r(h'_i) - 1), p(i, r(h'_i))$ . The initial sample  $S(P_i)$  is refined, in the way described,  $2 \log \log n - 2$  times. Somewhat surprisingly, what results is a good sample  $S(P)$  of  $P$ . The details follow.

### Algorithm Find-good-sample

**Step 1** For every  $i$ , ( $1 \leq i \leq m$ ), build sample  $S(P_i)$  by retaining every  $n^{\frac{1}{2}}$ -th item in  $P_i$ .

**Step 2** Repeat the following substeps  $2 \log \log n - 2$  times.

**Step 2.1** Compute the upper hull  $U(S(P))$  of  $S(P)$  and determine the left and right contact points  $p(i, g'_i)$  and  $p(i, h'_i)$  of  $S(P_i)$  with respect to  $U(S(P))$ . If  $S(P_i)$  has no contact point, determine the pseudo contact point  $p(i, g'_i)$  of  $S(P_i)$ .

**Step 2.2** Let  $p(i, l(g'_i))$  and  $p(i, r(g'_i))$  be, respectively, the left and right neighbor of  $p(i, g'_i)$  in  $S(P_i)$ ; similarly, let  $p(i, l(h'_i))$  and  $p(i, r(h'_i))$  be the left and right neighbor of  $p(i, h'_i)$  in  $S(P_i)$ . Update  $S(P_i)$  by the addition of:

- (1) every  $(g'_i - l(g'_i))^{\frac{1}{2}}$ -th item from the sequence  $\{p(i, l(g'_i)), p(i, l(g'_i) + 1), \dots, p(i, g'_i)\}$ ,
- (2) every  $(r(g'_i) - g'_i)^{\frac{1}{2}}$ -th item from the sequence  $\{p(i, g'_i), p(i, g'_i + 1), \dots, p(i, r(g'_i))\}$ ,
- (3) every  $(h'_i - l(h'_i))^{\frac{1}{2}}$ -th item from the sequence  $\{p(i, l(h'_i)), p(i, l(h'_i) + 1), \dots, p(i, h'_i)\}$ ,

- (4) every  $(r(h'_i) - h'_i)^{\frac{1}{2}}$ -th item from the sequence  $\{p(i, h'_i), p(i, h'_i + 1), \dots, p(i, r(h'_i))\}$ . If  $S(P_i)$  does not have distinct contact points, then only (1) and (2) are executed for the unique contact point or for the pseudo contact point  $p(i, g'_i)$ .

Let  $p(i, g_i)$  and  $p(i, h_i)$  be the left and right contact points (if any) of  $P_i$  with respect to  $U(P)$ . The span of  $p(i, g_i)$  with respect to  $S(P_i)$  is defined to be  $\min\{g_i - l(g_i), r(g_i) - g_i\}$  if  $p(i, g_i)$  belongs to  $S(P_i)$ ; otherwise, with  $p(i, f)$  standing for the (unique) point in  $S(P_i)$  for which  $l(f) < g_i < f$ , the span is  $f - l(f)$ . The span of  $p(i, h_i)$  is defined analogously. Intuitively, the span of point  $p(i, g_i)$  measures the "density" of  $S(P_i)$  around  $p(i, g_i)$ . Notice that  $S(P)$  is a good sample whenever the span of all the contact points of  $P$  is 1.

Clearly, the span of a contact point of  $P_i$  with respect to  $U(P)$  does not increase during the execution of the above algorithm, because no sample point is ever removed from  $S(P)$ . Somewhat surprisingly, the spans decrease quite fast. As we are about to show, when Find-good-sample terminates, the span of all the contact points is reduced to 1.

Fix an upper hull tangent  $p(i, h_i)p(j, g_j)$ , ( $i < j$ ), of  $U(P)$ . At the end of Step 1 the span of  $p(i, h_i)$  and  $p(j, g_j)$  is  $n^{\frac{1}{2}}$ . Consider what happens in the first iteration of Step 2. In Step 2.1 we determine the right contact point  $p(i, h'_i)$  of  $S(P_i)$  and the left contact point  $p(j, g'_j)$  of  $S(P_j)$  with respect to  $U(S(P))$ . Step 2.2 retains every  $n^{\frac{1}{2}}$ -th item from  $\{p(i, l(h'_i)), p(i, l(h'_i) + 1), \dots, p(i, r(h'_i))\}$  and every  $n^{\frac{1}{2}}$ -th item from  $\{p(j, l(g'_j)), p(j, l(g'_j) + 1), \dots, p(j, r(g'_j))\}$  and adds them to  $S(P_i)$  and  $S(P_j)$ , respectively. Lemma 3.3 guarantees that at least one of  $l(h'_i) < h_i < r(h'_i)$  or  $l(g'_j) < g_j < r(g'_j)$  holds. Thus, at least one of the spans of  $p(i, h_i)$  and  $p(j, g_j)$  decreases from  $n^{\frac{1}{2}}$  to  $n^{\frac{1}{4}}$ . More generally, we have the following result.

**Lemma 4.1** *In Step 2.1 of each of the  $2 \log \log n - 2$  iterations of Step 2, at least one of the following conditions is satisfied:*

- (a) the span  $s$ , ( $s > 1$ ), of  $p(i, h_i)$  decreases to  $s^{1/2}$ ,
- (b) the span  $s'$ , ( $s' > 1$ ), of  $p(j, g_j)$  decreases to  $s'^{1/2}$ ,
- (c) the span of both  $p(i, h_i)$  and  $p(j, g_j)$  is 1.

**Proof.** If  $s > 1$  and  $s' > 1$  hold, then Lemma 3.3 ensures that either (a) or (b) is satisfied. If exactly one of  $s > 1$  or  $s' > 1$  holds, say,  $s = 1$  and  $s' > 1$  holds, then Lemma 3.4 ensures that (b) occurs. If both  $s = 1$  and  $s' = 1$ , then clearly

(c) holds. Hence, in each iteration, Step 2.1 decreases at least one of the spans of  $p(i, h_i)$  and  $p(j, h_j)$  as claimed.  $\square$

Lemma 4.1 implies the main result of this section.

**Theorem 4.2** *When Find-good-sample terminates,  $S(P)$  is a good sample of  $P$ .*

**Proof.** Notice that condition (a) in Lemma 4.1 cannot hold true for more than  $\log \log n - 1$  iterations. Similarly, condition (b) cannot hold true for more than  $\log \log n - 1$  iterations. Therefore, at the end of  $2 \log \log n - 2$  iterations condition (c) must hold, as claimed.  $\square$

As it turns out, algorithm Find-good-sample has a number of additional properties that, collectively, allow us to implement each iteration in  $O(1)$  time on the reconfigurable mesh. The proofs will be given in Section 6.

- In each iteration at most  $4n^{1/4}$  new sample points are added to  $S(P_i)$ ,
- The new sample points are “sparse” in the sense that from any sequence  $p(i, f), p(i, f+1), \dots, p(i, f+s-1)$  of  $s$  points, at most  $4s^{1/2}$  points are added to  $S(P_i)$ .
- $S(P)$  contains at most  $mn^{1/2} + (2 \log \log n - 2) \cdot 4mn^{1/4} \leq 2mn^{1/2}$  sample points during the execution of algorithm.

## 5 Data movement and basic techniques

We begin by discussing a data movement technique that will be used time and again in the remainder of this work. Consider a set  $A = \{a(1), a(2), \dots, a(n)\}$  of  $n = 2^d \times 2^d$  items, with some of the items marked. The set  $A$  is pretiled, in proximity order, on a reconfigurable mesh  $\mathcal{M}$  of size  $2^d \times 2^d$ . We are interested in identifying, for every item in  $A$ , the *next marked item* in proximity order. More precisely, we wish to identify a set  $A' = \{a'(1), a'(2), \dots, a'(n)\}$  such that  $a'(i) = a(j)$  with  $j = \min\{k \mid k > i \text{ and } a(k) \text{ is marked}\}$ . If no  $a(k)$  with  $k > i$  is marked, then  $a'(i)$  is  $\emptyset$ . The task at hand involves (1) establishing a bus embedding of the proximity order as shown in Figure 2, (2) having each marked item segment the bus, and (3) mandating each marked item to broadcast its identity on the sub-bus containing items preceding it in proximity order. Thus, we have

**Lemma 5.1** *Given a set of  $n = 2^d \times 2^d$  items pretiled on a reconfigurable mesh of the same size*

*with some of the items marked, the task of identifying for each item the next marked item in proximity order can be performed in  $O(1)$  time.*

In essentially the same way one can identify the *previous marked item*, and the *leftmost marked item*, that is, the first marked item in proximity order.

Next, suppose that the set  $A$  is partitioned into  $2^{2(d-d')}$  subsets  $A(1), A(2), \dots, A(2^{2(d-d')})$  of size  $2^{d'} \times 2^{d'}$  each. Partition  $\mathcal{M}$  into horizontal and vertical strips of size  $2^{d'} \times 2^d$  and  $2^d \times 2^{d'}$ , respectively. For  $1 \leq i, j \leq 2^{d-d'}$ , let  $M_{i,j}$  be the submesh of  $\mathcal{M}$  determined by the intersection of the  $i$ -th horizontal and the  $j$ -th vertical strip. Each  $A(k)$  is stored in one of the submeshes  $M_{i,j}$ .

The operation Move-leftmost-items is to move the leftmost marked item in each  $M_{i,j}$  to the first row of  $\mathcal{M}$ . To ensure that we have enough bandwidth to complete the data movement in  $O(1)$  time, we insist that  $2^d \geq 2^{2(d-d')}$  or, equivalently, that  $d \leq 2d'$ . Due to the strong page limitation, we will omit the detail of Move-leftmost-items. We have the following result.

**Lemma 5.2** *Given a reconfigurable mesh of size  $2^d \times 2^d$  partitioned into non-overlapping submeshes of size  $2^{d'} \times 2^{d'}$ , with  $\frac{d}{2} \leq d' < d$ , the task of moving the leftmost marked item in each submesh to the first row of the platform can be performed in  $O(1)$  time.*

Suppose that each submesh  $M_{i,j}$  of size  $2^{d'} \times 2^{d'}$  contains  $2^{d''}$  items stored in the leftmost  $2^{d''}$  positions of its top row. The goal of the operation Move-to-top-row is to move all these items to the top row of  $\mathcal{M}$ . Note that the total number of items is  $2^{d-d'} \times 2^{d-d'} \times 2^{d''} = 2^{2d-2d'+d''}$  and the number of processors in the top row is  $2^d$ . In order to ensure that no processor receives more than one item (i.e.  $2^{2d-2d'+d''} \leq 2^d$ ), we insist that  $d \leq 2d' - d''$ .

We omit the detail of Move-to-top-row due to the page limitation.

**Lemma 5.3** *Consider a reconfigurable mesh of size  $2^d \times 2^d$  partitioned into non-overlapping submeshes of size  $2^{d'} \times 2^{d'}$  and assume that each submesh has at most  $2^{d''}$  items in its top row. The task of moving all the items to the first row of the platform can be performed in  $O(1)$  time, provided that  $d + d'' \leq 2d'$ .*

## 6 An $O((\log \log n)^2)$ -time upper hull algorithm

The main goal of this section is to show how all the pieces fit together to yield an  $O((\log \log n)^2)$

time convex hull algorithm on a reconfigurable mesh  $\mathcal{M}$  of size  $2^d \times 2^d$ , where  $d \geq 2$ .

Let  $P = \{p(1), p(2), \dots, p(n)\}$  be a set of  $n = 2^{2d}$  points sorted by  $x$ -coordinate and pretiled on  $\mathcal{M}$  in proximity order. Write  $d_0 = \lceil \frac{2d}{3} \rceil$  and let  $d_s = \max\{\lceil d_{s-1} - \frac{d}{8} \rceil, \lceil \frac{d_{s-1}}{2} \rceil\}$ , for all  $s \geq 1$ . It is easy to see that both  $d_{s-1} - d_s \leq \lfloor \frac{d}{8} \rfloor$  and  $d_s \geq \lceil \frac{d_{s-1}}{2} \rceil$  hold. Further, let  $T$  be the smallest integer for which  $d_T = 1$ . Clearly,  $T \in O(\log d) = O(\log \log n)$ . Partition  $P$  into  $2^{2(d-d_0)}$  subsets  $A(1), A(2), \dots, A(2^{2(d-d_0)})$ , such that each  $A(i)$  involves  $2^{2d_0}$  points consecutive in proximity order stored in a submesh  $M(i)$  of  $\mathcal{M}$  of size  $2^{d_0} \times 2^{d_0}$ .

For each  $i_0$ , ( $1 \leq i_0 \leq 2^{2(d-d_0)}$ ), the set  $A(i_0)$  is partitioned into  $2^{2(d_0-d_1)}$  subsets  $A(i_0, 1), A(i_0, 2), \dots, A(i_0, 2^{2(d_0-d_1)})$ , with each  $A(i_0, j)$ , ( $1 \leq j \leq 2^{2(d_0-d_1)}$ ), consisting of  $2^{2d_1}$  points consecutive in proximity order, stored in the submesh  $M(i_0, j)$  of  $M(i_0)$  of size  $2^{d_1} \times 2^{d_1}$ . Proceedings in this way, each set  $A(i_0, i_1, \dots, i_{s-1})$ , ( $1 \leq s \leq T$ ), containing  $2^{2d_{s-1}}$  points and stored in proximity order in the submesh  $M(i_0, i_1, \dots, i_{s-1})$  of size  $2^{d_{s-1}} \times 2^{d_{s-1}}$  is partitioned into  $2^{2(d_{s-1}-d_s)}$  subsets  $A(i_1, i_2, \dots, i_{s-1}, j)$ , ( $1 \leq j \leq 2^{2(d_{s-1}-d_s)}$ ), with each  $A(i_1, i_2, \dots, i_{s-1}, j)$  involving  $2^{2d_s}$  points consecutive in proximity order, stored in a submesh  $M(i_1, i_2, \dots, i_{s-1}, j)$  of size  $2^{d_s} \times 2^{d_s}$ . To simplify the exposition, in the remainder of this section we shall use sets of points and the submeshes containing them interchangeably. This should create no ambiguity.

#### Algorithm Compute-upper-hull

**Step 1** Recursively, determine the upper hull  $P(i_0)$  of the points in each submesh  $M(i_0)$ , ( $1 \leq i_0 \leq 2^{2(d-d_0)}$ ), and mark all the points in  $P(i_0)$ ;

**Step 2** In each  $M(i_0)$ , ( $1 \leq i_0 \leq 2^{2(d-d_0)}$ ), extract a sample  $S(P(i_0))$  of  $P(i_0)$  by retaining the leftmost marked point in each submesh  $M(i_0, i_1)$ , ( $1 \leq i_1 \leq 2^{2(d_0-d_1)}$ ). Add to  $S(P(i_0))$  the rightmost marked point in  $M(i_0)$ . Move the points in  $S(P) = S(P(1)) \cup S(P(2)) \cup \dots \cup S(P(2^{2(d-d_0)}))$  to the top row of  $\mathcal{M}$ . Using an  $O(1)$ -time prefix sums algorithm move  $S(P)$  to the leftmost  $|S(P)|$  positions of the top row of  $\mathcal{M}$ ;

**Step 3** Repeat the following substeps  $2T$  times:

**Step 3.1** Using one of the existing algorithms for the sparse case [3, 6], compute the upper hull  $U(S(P))$  of  $S(P)$  and determine the left and right contact points  $p(g_i)$  and  $p(h_i)$ , if any, of  $S(P(i))$  with respect to  $U(S(P))$ . If  $S(P(i))$  does not have contact points, then find the pseudo contact

point  $p(g_i)$  of  $S(P(i))$ , using an  $O(1)$ -time minimum algorithm;

**Step 3.2** Let  $p(l(g_i))$  and  $p(r(g_i))$  be, respectively, the left and right neighbors of  $p(g_i)$  in  $S(P(i))$ . If the two points  $p(l(g_i))$  and  $p(g_i)$  are in the same submesh  $M(i_0, i_1, \dots, i_T)$  of size  $2^1 \times 2^1$ , then add to  $S(P(i))$  the remaining two points  $P(i_0, i_1, \dots, i_T) - \{p(l(g_i)), p(g_i)\}$  in  $M(i_0, i_1, \dots, i_T)$ . Otherwise, let  $M(i_0, i_1, \dots, i_{s-1}, i_s)$ , ( $s > T$ ), be the smallest submesh containing both  $p(g_i)$  and  $p(l(g_i))$ . Add to  $S(P(i))$  the leftmost marked point in each of the  $M(i_0, i_1, \dots, i_{s-1}, i_s, j)$ , ( $1 \leq j \leq 2^{2(d_s-d_{s+1})}$ ). Repeat the procedure for the three pairs of points  $p(g_i)$  and  $p(r(g_i))$ ,  $p(l(h_i))$  and  $p(h_i)$ , and  $p(h_i)$  and  $p(r(h_i))$ . Move all newly selected sample points to the top row of  $\mathcal{M}$ . Using prefix sums in the obvious way, the updated  $S(P)$  is stored, one item per processor, in the leftmost  $|S(P)|$  positions of the top row of  $\mathcal{M}$ ;

**Step 4** Compute the upper hull  $U(S(P))$  of  $S(P)$  and determine the left and the right contact points  $p(g_i)$  and  $p(h_i)$  of  $S(P(i))$ , ( $1 \leq i \leq 2^{2(d-d_0)}$ ), with respect to  $U(S(P))$ . Retain all the points in  $P(i)$  between  $p(g_i)$  and  $p(h_i)$  (inclusive) as the upper hull points of  $P$ .

The correctness of the algorithm follows immediately by the induction hypothesis along with Theorem 4.2. To argue for the complexity, note that Step 2 as well as each iteration of Step 3 of the algorithm can be implemented in  $O(1)$  time. In Step 2, Move-leftmost-items is performed in every submesh  $M(i_0)$ , ( $1 \leq i_0 \leq 2^{2(d-d_0)}$ ), to move the leftmost marked point in  $M(i_0, i_1)$  ( $1 \leq i_1 \leq 2^{2(d_0-d_1)}$ ), to the top row of  $M(i_0)$ . Since  $d_1 \geq \lceil \frac{d_0}{2} \rceil$ , Lemma 5.2 guarantees that this can be done in  $O(1)$  time. Since  $d_0 - d_1 \leq \lfloor \frac{d}{8} \rfloor$ , the top row of  $M(i_0)$  contains at most  $2^{2(d_0-d_1)} \leq 2^{\lfloor \frac{d}{8} \rfloor} + 1$  marked points. Further, the points thus obtained are moved to the top row of  $\mathcal{M}$ . Since  $2d_0 - 2(d_0 - d_1) = 2\lceil \frac{2d}{3} \rceil - \lfloor \frac{d}{4} \rfloor \geq d$ , Lemma 5.3 guarantees that this data movement operation can be performed in  $O(1)$  time. Further, the rightmost marked point of each  $M(i_0)$  can be moved to the top row of  $\mathcal{M}$  in  $O(1)$  time by executing Move-leftmost-items.

In the first iteration of Step 3.1,  $S(P)$  contains at most  $2^{2(d-d_0)} \cdot (2^{\lfloor \frac{d}{4} \rfloor} + 1) \leq 2^{2\lfloor \frac{d}{4} \rfloor} \cdot (2^{\lfloor \frac{d}{4} \rfloor} + 1) < 2^{\lfloor \frac{11d}{4} \rfloor} + 2^{\lfloor \frac{3d}{4} \rfloor} \leq 2^d$  points. Thus, the upper hull of  $S(P)$  can be computed in  $O(1)$  time. In Step 3.2, in case the points  $p(l(g_i))$  and  $p(g_i)$  belong to the same submesh  $M(i_0, i_1, \dots, i_T)$ , the remaining two points in the submesh are selected and moved to the top row of  $M(i_0)$ .

Clearly, this can be done in  $O(1)$  time. Otherwise, Move-leftmost-items is executed, in parallel, in each  $M(i_0, i_1, \dots, i_{s-1}, i_s)$  to move the leftmost marked item in  $M(i_0, i_1, \dots, i_{s-1}, i_s, i_{s+1})$ , ( $1 \leq i_{s+1} \leq 2^{2d_{s+1}}$ ), to the top row of  $M(i_0, i_1, \dots, i_{s-1}, i_s)$ . Since  $d_{s+1} \geq \lceil \frac{d_s}{2} \rceil$  holds, Lemma 5.2 guarantees that this can be done in  $O(1)$  time. Further, the leftmost marked items are moved to the top row of  $M(i_0)$ . The same task for the other three pairs  $p(g_i)$  and  $p(r(g_i))$ ,  $p(l(g_i))$  and  $p(g_i)$ , and  $p(h_i)$  and  $p(r(h_i))$  is performed in  $O(1)$  time. Next, these newly selected sample points are moved to the top row of  $\mathcal{M}$  by using procedure Move-to-top-row. Since  $d_s - d_{s+1} \leq \lfloor \frac{d}{8} \rfloor$ , at most  $4 \cdot 2^{2(d_s - d_{s+1})} \leq 2^{\lfloor \frac{d}{4} \rfloor + 2}$  points are moved. Since  $2d_0 - (\lfloor \frac{d}{4} \rfloor + 2) = 2\lceil \frac{2d}{3} \rceil - \lfloor \frac{d}{4} \rfloor - 2 \geq d$  for sufficiently large  $d$ , Lemma 5.3 guarantees that this data movement can be performed in  $O(1)$  time.

In each iteration, at most  $4 \cdot 2^{\lfloor \frac{d}{4} \rfloor}$  sample points are added to each  $S(P_i)$ . Therefore, at most  $2^{2(d-d_0)} \cdot 2^{\lfloor \frac{d}{4} \rfloor} \leq 4 \cdot 2^{\lfloor \frac{11d}{4} \rfloor}$  points are added to  $S(P)$  in each execution of Step 3.2. Since Step 3.2 is executed  $2T$  times,  $S(P)$  has at most  $2^{\lfloor \frac{11d}{4} \rfloor} + 2^{\lfloor \frac{2d}{3} \rfloor} + 2T \cdot 4 \cdot 2^{\lfloor \frac{11d}{4} \rfloor} < 2^d$  points, for sufficiently large  $d$ . Thus, the upper hull of  $S(P)$  can be computed in  $O(1)$  time.

Since each substep of Step 3 can be performed in  $O(1)$  time, it follows that Step 3 takes, altogether,  $O(T) = O(\log \log n)$  time. Since the depth of recursion is  $O(\log \log n)$ , the overall running time of algorithm Compute-upper-hull is bounded by  $O((\log \log n)^2)$ . Thus we have

**Lemma 6.1** *The upper hull of  $n$  points sorted by  $x$ -coordinate and stored in proximity order on a reconfigurable mesh of size  $n = 2^d \times 2^d$  can be computed in  $O((\log \log n)^2)$  time.*

Due to the page limitation, we omit work-optimize the algorithm Compute-upper-hull. The journal version of this paper will present the following result.

**Theorem 6.2** *The upper hull of  $n$  points sorted by  $x$ -coordinate and stored in proximity order,  $(\log \log n)^2$  points per processor, in a reconfigurable mesh of size  $\frac{\sqrt{n}}{\log \log n} \times \frac{\sqrt{n}}{\log \log n}$  can be computed in  $O((\log \log n)^2)$  time.*

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