# M凸関数最小化問題に対する領域スケーリング法

田村 明久
京都大学 数理解析研究所
tamura@kurims.kyoto-u.ac.jp

要旨 近年, 室田により提案された離散凸解析においてM凸関数は中心的役割を果たす. M凸関数最小化問題に対する多項式時間スケーリング法を提案する.

# A Domain Scaling Algorithm for M-convex Function Minimization

Akihisa Tamura
Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
tamura@kurims.kyoto-u.ac.jp

**Abstract** The concept of M-convex functions plays a central role in "discrete convex analysis," a unified framework of discrete optimization introduced by Murota. We develop a polynomial time scaling algorithm for M-convex function minimization problem.

## 1 Introduction

The concept of M-convexity was introduced by Murota [14, 15] in the context of "discrete convex analysis" which is a unified framework of discrete optimization with reference to existing studies on submodular functions [4, 8, 11], generalized polymatroids [5, 6, 8], valuated matroids [2, 3, 16] and convex analysis [20]. M-convex functions have applications in the areas of mathematical economics, engineering and so on [1, 17, 19]. This paper addresses a fundamental problem, namely, M-convex function minimization problem, which is an extension of separable discrete convex function minimization over an integral base polyhedron [7, 9, 10].

Let $V$ be a nonempty finite set of cardinality $n$, and $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_{++}$ be the sets of reals, integers and positive integers, respectively. Given a vector $z = (z(v) : v \in V) \in \mathbf{Z}^V$, its positive support and negative support are defined by

$$\mathrm{supp}^+(z) = \{v \in V \mid z(v) > 0\} \quad \text{and} \quad \mathrm{supp}^-(z) = \{v \in V \mid z(v) < 0\}.$$

The characteristic vector (unit vector) of $v \in V$ is denoted by $\chi_v$. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *effective domain* of $f$ is defined by

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M-convex* [14, 15] if it satisfies

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ :

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Several algorithms for minimizing an M-convex function $f$ are known. Since a locally minimal solution of $f$ is globally minimal, a descent algorithm finds an optimal solution, but it does not terminate in polynomial time. Shioura [21] showed a key property which guarantees that any nonoptimal solution in the effective domain can be efficiently separated from a minimizer, and exploited this property to develop the domain reduction algorithm, the first polynomial time algorithm for M-convex function minimization. Time complexity of the domain reduction algorithm is $O(Fn^4(\log L)^2)$, where $F$ is an upper bound of the time to evaluate $f$ and $L = \max\{|x(u) - y(u)| \mid u \in V,\ x, y \in \mathrm{dom}\, f\}$. Moriguchi, Murota and Shioura [12, 13] showed the proximity property and proposed a scaling technique. They showed that the technique gives an $O(Fn^3 \log(L/n))$-time algorithm for a subclass of M-convex functions $f$ such that "scaled" function

$$f_\alpha(x) = f(x^0 + \alpha x) \qquad (x \in \mathbf{Z}^V)$$

is also M-convex for any $x^0 \in \mathrm{dom}\, f$ and $\alpha \in \mathbf{Z}_{++}$. Although their algorithm, being a descent algorithm, works for any M-convex function, it does not terminate in polynomial time in general. This is because M-convexity is not inherited in the scaling operation, that is, $f_\alpha$ is not necessarily M-convex.

We develop an $O(Fn^3 \log(L/n))$-time scaling algorithm applicable to any M-convex function. The novel idea of the algorithm is to use an individual scaling factor $\alpha_v \in \mathbf{Z}_{++}$ for each $v \in V$.

## 2 Properties of an M-convex Function

Here we review known results which are utilized in the next section.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function and $\arg\min f$ denote the set of all minimizers of $f$, that is,

$$\arg\min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y)\ (\forall y \in \mathbf{Z}^V)\}.$$

By (M-EXC), $\mathrm{dom}\, f$ lies on a hyperplane whose normal vector is the vector of all ones, that is, for any $x, y \in \mathrm{dom}\, f$, $x(V) = y(V)$ holds, where $x(V) = \sum_{v \in V} x(v)$. It is also known that $\arg\min f$ and $\mathrm{dom}\, f$ for an M-convex function $f$ are an M-convex set, where a nonempty set $B \subseteq \mathbf{Z}^V$ is said to be an *M-convex set* if it satisfies

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ :

$$x - \chi_u + \chi_v \in B, \quad y + \chi_u - \chi_v \in B.$$

An M-convex set is a synonym of the set of integer points of the base polyhedron of an integral submodular system (see [8] for submodular systems).

**Lemma 2.1 ([14, 15])** *For an M-convex function $f$, $\mathrm{dom}\, f$ and $\arg\min f$ are M-convex sets.*

The following property is a direct consequence of Lemma 2.1.

**Proposition 2.2** *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function. For $u, v \in V$ and $\gamma_1, \gamma_2 \in \mathbf{Z}$, suppose that there exist $x_1, x_2 \in \arg\min f$ with $x_1(u) \leq \gamma_1$ and $x_2(v) \geq \gamma_2$, where $\gamma_2 \leq \gamma_1$ if $u = v$. Then, there exists $x_* \in \arg\min f$ with $x_*(u) \leq \gamma_1$ and $x_*(v) \geq \gamma_2$.*

M-convex functions have various features of convex functions. For example, a locally minimal solution of an M-convex function is globally minimal.

**Theorem 2.3 ([14, 15])** *For an M-convex function $f$ and $x \in \mathrm{dom}\, f$, $f(x) \leq f(y)$ for any $y \in \mathbf{Z}^V$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$.*

The following properties, Theorems 2.4 and 2.5, play important roles in our algorithm. The former is due to Shioura [21] and the latter is implicit in [12, 13].

**Theorem 2.4 ([21])** *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function with* $\arg\min f \neq \emptyset$. *For* $x \in \mathrm{dom}\, f$ *and* $v \in V$, *the following statements hold.*

(a) *If* $f(x - \chi_v + \chi_u) = \min_{s \in V} f(x - \chi_v + \chi_s)$, *then there exists* $x_* \in \arg\min f$ *such that* $x(u) - \chi_v(u) + 1 \leq x_*(u)$.

(b) *If* $f(x + \chi_v - \chi_u) = \min_{s \in V} f(x + \chi_v - \chi_s)$, *then there exists* $x_* \in \arg\min f$ *such that* $x_*(u) \leq x(u) + \chi_v(u) - 1$.

**Theorem 2.5** *Let* $f$ *be an M-convex function with* $\arg\min f \neq \emptyset$. *For* $x \in \mathrm{dom}\, f$, $v \in V$ *and* $\alpha \in \mathbf{Z}_{++}$, *the following statements hold.*

(a) *If* $f(x) = \min_{s \in V} f(x - \alpha(\chi_v - \chi_s))$, *then there exists* $x_* \in \arg\min f$ *such that* $x(v) - (n-1)(\alpha - 1) \leq x_*(v)$.

(b) *If* $f(x) = \min_{s \in V} f(x + \alpha(\chi_v - \chi_s))$, *then there exists* $x_* \in \arg\min f$ *such that* $x_*(v) \leq x(v) + (n-1)(\alpha - 1)$.

Here we show a new property, Theorem 2.6 below, as a common generalization of Theorems 2.4 and 2.5. A special case of Theorem 2.6 with $\alpha = 1$ is equivalent to Theorem 2.4. Another special case of Theorem 2.6 with $u = v$ is identical with Theorem 2.5.

**Theorem 2.6** *Let* $f$ *be an M-convex function with* $\arg\min f \neq \emptyset$. *For* $x \in \mathrm{dom}\, f$, $v \in V$ *and* $\alpha \in \mathbf{Z}_{++}$, *the following statements hold.*

(a) *If* $f(x - \alpha(\chi_v - \chi_u)) = \min_{s \in V} f(x - \alpha(\chi_v - \chi_s))$, *then there exists* $x_* \in \arg\min f$ *such that* $x(u) - \alpha(\chi_v(u) - 1) - (n-1)(\alpha - 1) \leq x_*(u)$.

(b) *If* $f(x + \alpha(\chi_v - \chi_u)) = \min_{s \in V} f(x + \alpha(\chi_v - \chi_s))$, *then there exists* $x_* \in \arg\min f$ *such that* $x_*(u) \leq x(u) + \alpha(\chi_v(u) - 1) + (n-1)(\alpha - 1)$.

**Proof.** Here we prove the assertion (a) because we can similarly prove (b). It is sufficient to consider the case where there exists $x_* \in \arg\min f$ such that $x_*(u)$ is maximum. Let $\widehat{x} = x - \alpha(\chi_v - \chi_u)$. Assume that $\widehat{x}(u) > x_*(u)$ and $k = \widehat{x}(u) - x_*(u)$.

CLAIM A: There exist $w_1, w_2, \ldots, w_k \in V \setminus \{u\}$ and $y_0 (= \widehat{x}), y_1, \ldots, y_k \in \mathrm{dom}\, f$ such that $y_i = y_{i-1} - \chi_u + \chi_{w_i}$ and $f(y_i) < f(y_{i-1})$ for $i = 1, 2, \ldots, k$.

[Proof of Claim A] Let $y_{i-1} \in \mathrm{dom}\, f$. By (M-EXC), for $y_{i-1}, x_*$ and $u \in \mathrm{supp}^+(y_{i-1} - x_*)$, there exists $w_i \in \mathrm{supp}^-(y_{i-1} - x_*) \subseteq V \setminus \{u\}$ such that

$$f(x_*) + f(y_{i-1}) \geq f(x_* + \chi_u - \chi_{w_i}) + f(y_{i-1} - \chi_u + \chi_{w_i}).$$

Since $f(x_*) < f(x_* + \chi_u - \chi_{w_i})$, we have $f(y_{i-1}) > f(y_{i-1} - \chi_u + \chi_{w_i}) = f(y_i)$.

CLAIM B: For any $w \in V \setminus \{u\}$ with $y_k(w) > \widehat{x}(w)$ and $\beta \in \mathbf{Z}$ with $0 \leq \beta \leq y_k(w) - \widehat{x}(w) - 1$, $f(\widehat{x} - (\beta + 1)(\chi_u - \chi_w)) < f(\widehat{x} - \beta(\chi_u - \chi_w))$ holds.

[Proof of Claim B] We prove the claim by induction on $\beta$. For $\beta$ with $0 \leq \beta \leq y_k(w) - \widehat{x}(w) - 1$, put $x' = \widehat{x} - \beta(\chi_u - \chi_w)$ and assume $x' \in \mathrm{dom}\, f$. Let $j_*$ $(1 \leq j_* \leq k)$ be the maximum index with $w_{j_*} = w$. Since $y_{j_*}(w) = y_k(w) > x'(w)$ and $\mathrm{supp}^-(y_{j_*} - x') = \{u\}$, we have $f(x') + f(y_{j_*}) \geq f(x' - \chi_u + \chi_w) + f(y_{j_*} + \chi_u - \chi_w)$ by (M-EXC). Claim A guarantees that $f(y_{j_*-1}) = f(y_{j_*} + \chi_u - \chi_w) > f(y_{j_*})$, and hence, $f(x') > f(x' - \chi_u + \chi_w)$.

The hypothesis of (a) and Claim B imply $\mu_w = y_k(w) - \widehat{x}(w) \leq \alpha - 1$ for any $w \in V \setminus \{u\}$, because

$$f(\widehat{x} - \mu_w(\chi_u - \chi_w)) < \cdots < f(\widehat{x} - (\chi_u - \chi_w)) < f(\widehat{x}) \leq f(\widehat{x} - \alpha(\chi_u - \chi_w))$$

holds for any $w$ with $\mu_w > 0$. Thus, we have

$$\widehat{x}(u) - x_*(u) = \widehat{x}(u) - y_k(u) = \sum_{w \in V \setminus \{u\}} \{y_k(w) - \widehat{x}(w)\} \leq (n-1)(\alpha - 1),$$

where the second equality follows from $\widehat{x}(V) = y_k(V)$. ∎

# 3 Proposed Algorithm

In this section, we describe a scaling algorithm of time complexity $O(Fn^3 \log(L/n))$ for the M-convex function minimization. It is assumed that the effective domain of a given M-convex function $f$ is bounded and that a vector $x^0 \in \mathrm{dom}\, f$ is given.

We preliminarily show that $L = \max\{|x(u) - y(u)| \mid u \in V,\ x, y \in \mathrm{dom}\, f\}$ can be computed in $O(Fn^2 \log L)$ time. For $x \in \mathrm{dom}\, f$ and $u, v \in V$, the exchange capacity associated with $x$, $u$ and $v$ is defined as

$$\tilde{c}_f(x, v, u) = \max\{\alpha \mid x + \alpha(\chi_v - \chi_u) \in \mathrm{dom}\, f\},$$

which can be computed in $O(F \log L)$ time by the binary search because $0 \leq \tilde{c}_f(x, v, u) \leq L$. For each $w \in V$, define

$$l_f(w) = \min\{x(w) \mid x \in \mathrm{dom}\, f\}, \quad u_f(w) = \max\{x(w) \mid x \in \mathrm{dom}\, f\}.$$

The values $l_f(w)$ and $u_f(w)$ can be calculated by the following algorithm in $O(Fn \log L)$ time.

**function** CALCULATE_BOUND$(f, x, w)$
  **input:** $f$ : M-convex function, $x \in \mathrm{dom}\, f$, $w \in V$ ;
  **output:** $(l_f(w), u_f(w))$ ;

$\ell 1$: number $V \setminus \{w\}$ from $v_2$ to $v_n$ ;
$\ell 2$: $y := z := x$ ;
$\ell 3$: **for** $i := 2$ **to** $n$ **do** { $y := y + \tilde{c}(y, v_i, w)$, $z := z + \tilde{c}(z, w, v_i)$ } ;
$\ell 4$: **return** $(y(w), z(w))$.

The correctness of the above algorithm can be verified easily from the fact that $\mathrm{dom}\, f$ satisfies (B-EXC) (see also [8, 21]).

**Lemma 3.1** *Values $l_f(w)$ and $u_f(w)$ for a fixed $w \in V$ can be computed in $O(Fn \log L)$ time, and $L$ in $O(Fn^2 \log L)$ time.*

For any two vectors $a, b \in \mathbf{Z}^V$, let $[a, b]$ denote the set $\{x \in \mathbf{Z}^V \mid a \leq x \leq b\}$ and $f_a^b$ be defined by

$$f_a^b(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ +\infty & \text{otherwise.} \end{cases}$$

Condition (M-EXC) directly yields the next property.

**Proposition 3.2** *For an M-convex function $f$ and $a, b \in \mathbf{Z}^V$, if $\mathrm{dom}\, f_a^b \neq \emptyset$ then $f_a^b$ is also M-convex.*

We go on to the main topic of describing our algorithm for minimizing $f$. The novel idea of the algorithm is to use an individual scaling factor $\alpha_v \in \mathbf{Z}_{++}$ for each $v \in V$. Besides the factors $\alpha_v$ ($v \in V$), the algorithm maintains a current vector $x \in \mathrm{dom}\, f$, two vectors $a, b \in \mathbf{Z}^V$ and a subset $V' \subseteq V$, and it preserves the following four conditions:

(c1) $x \in \mathrm{dom}\, f \cap [a, b] = \mathrm{dom}\, f_a^b$,

(c2) $b(v) - a(v) \le 2n\alpha_v$ for $v \in V$,

(c3) $\arg\min f \cap [a, b] \ne \emptyset$, (i.e., $\arg\min f_a^b \subseteq \arg\min f$),

(c4) there exists $x_* \in \arg\min f$ such that $x_*(v) = x(v)$ for all $v \in V \setminus V'$.

These parameters are initially put as $x := x^0$, $V' := V$, $a(v) := l_f(v)$, $b(v) := u_f(v)$ and $\alpha_v := 2^{\lceil \log_2 \{(u_f(v) - l_f(v))/n\} \rceil}/2$ for $v \in V$. Thus, conditions (c1) to (c4) are initially satisfied. At each iteration of the algorithm, interval $[a, b]$ is strictly reduced and $V'$ is not increased. By (c4) and the fact that $y(V)$ is a constant for any $y \in \mathrm{dom}\, f$, the algorithm stops if $|V'| \le 1$; then the current $x$ is a minimizer of $f$. The algorithm terminates in $O(n^2 \log(L/n))$ iterations and requires $O(Fn)$ time at each iteration. Hence, the total time complexity of the algorithm is $O(Fn^3 \log(L/n))$.

Before entering into a precise description of the algorithm, we briefly explain its typical behaviour. First, take $v \in V'$ arbitrarily, and find $u \in V'$ minimizing $f_a^b(x - \chi_u + \chi_v)$. Here we explain what the algorithm does in the case where $u \ne v$. Then, there exists $x_1 \in \arg\min f_a^b$ with $x_1(u) \le x(u) - 1$ by (b) of Theorem 2.4. This inequality suggests that an optimal solution can be found by decreasing $x(u)$. Next find $w \in V'$ minimizing $f_a^b(x - \alpha_u(\chi_u - \chi_w))$. When $u$ can be chosen as $w$ attaining the minimum value, (a) of Theorem 2.5 guarantees that there exists $x_2 \in \arg\min f_a^b$ with $x(u) - (n-1)(\alpha_u - 1) \le x_2(u)$. By Proposition 2.2, there exists $x_* \in \arg\min f_a^b$ with $x(u) - (n-1)(\alpha_u - 1) \le x_*(u) \le x(u) - 1 < x(u)$. Thus, we can put $a(u) := \max[a(u), x(u) - (n-1)(\alpha_u - 1)]$ and $b(u) := x(u)$. Since $b(u) - a(u) \le n\alpha_u$, the scaling factor $\alpha_u$ can be divided by 2 without violating (c2). In the other case with $w \ne u$, we update $a$, $b$ and $x$ as $b(u) := x(u) - 1$, $x := x - \alpha_u(\chi_u - \chi_w)$ and $a(w) := \max[a(w), x(w) - (n-1)(\alpha_u - 1)]$, where the update of $a$ is justified by (a) of Theorem 2.6. This is a part of our algorithm described below (see CASE2).

A complete description of our algorithm is now given.

**algorithm** COORDINATEWISE_SCALING$(f, V, x^0)$
  **input:** $f$ : M-convex function with bounded $\mathrm{dom}\, f \subset \mathbf{Z}^V$, $x^0 \in \mathrm{dom}\, f$ ;
  **output:** a minimizer of $f$ ;

ℓ01: $n := |V|$, $V' := V$, $x := x^0$ ;
ℓ02: **for each** $v \in V$ **do** $(a(v), b(v)) :=$ CALCULATE_BOUND$(f, x, v)$ ;
ℓ03: **for each** $v \in V$ **do** $\alpha_v := 2^{\lceil \log_2 \{(b(v) - a(v))/n\} \rceil}/2$ ;
ℓ04: **while** $|V'| \ge 2$ **do** {
ℓ05: take $v \in V'$ ;
ℓ06: find $u_1 \in V'$ : $f_a^b(x + \chi_v - \chi_{u_1}) = \min_{s \in V' \setminus \{v\}} f_a^b(x + \chi_v - \chi_s)$ ;
ℓ07: find $u_2 \in V'$ : $f_a^b(x - \chi_v + \chi_{u_2}) = \min_{s \in V' \setminus \{v\}} f_a^b(x - \chi_v + \chi_s)$ ;
ℓ08: **if** $f_a^b(x) \le f_a^b(x + \chi_v - \chi_{u_1})$ **and** $f_a^b(x) \le f_a^b(x - \chi_v + \chi_{u_2})$ **then**
ℓ09: { $a(v) := b(v) := x(v)$, $V' := V' \setminus \{v\}$ } ;
ℓ10: **else if** $f_a^b(x) > f_a^b(x + \chi_v - \chi_{u_1})$ **then** CASE2$(u_1)$ ;
ℓ11: **else** $(f_a^b(x) > f_a^b(x - \chi_v + \chi_{u_2}))$ **then** CASE3$(u_2)$ ;

$\ell$12: } ;
$\ell$13: **return** $(x)$ ;

**function** CASE2$(u)$  $(\exists x_1 \in \arg\min f$ with $x_1(u) \le x(u) - 1)$
$\ell$1: $W_u := \arg\min_{s \in V'} f_a^b(x - \alpha_u(\chi_u - \chi_s))$ ;
$\ell$2: **if** $u \in W_u$ **then**
$\ell$3: $\{\ a(u) := \max[a(u), x(u) - (n-1)(\alpha_u - 1)], \quad b(u) := x(u)\ \}$ ;
$\ell$4: **else** $(u \notin W_u)$ {
$\ell$5: take $w \in W_u$, $b(u) := x(u) - 1$, $x := x - \alpha_u(\chi_u - \chi_w)$ ;
$\ell$6: $a(w) := \max[a(w), x(w) - (n-1)(\alpha_u - 1)]$ ;
$\ell$7: UPDATE_FACTOR$(w)$ } ;
$\ell$8: UPDATE_FACTOR$(u)$ ;
$\ell$9: **return** ;

**function** CASE3$(u)$  $(\exists x_2 \in \arg\min f$ with $x_2(u) \ge x(u) + 1)$
$\ell$1: $W_u := \arg\min_{s \in V'} f_a^b(x + \alpha_u(\chi_u - \chi_s))$ ;
$\ell$2: **if** $u \in W_u$ **then**
$\ell$3: $\{\ a(u) := x(u), \quad b(u) := \min[b(u), x(u) + (n-1)(\alpha_u - 1)]\ \}$ ;
$\ell$4: **else** $(u \notin W_u)$ {
$\ell$5: take $w \in W_u$, $a(u) := x(u) + 1$, $x := x + \alpha_u(\chi_u - \chi_w)$ ;
$\ell$6: $b(w) := \min[b(w), x(w) + (n-1)(\alpha_u - 1)]$ ;
$\ell$7: UPDATE_FACTOR$(w)$ } ;
$\ell$8: UPDATE_FACTOR$(u)$ ;
$\ell$9: **return** ;

**function** UPDATE_FACTOR$(s)$
$\ell$1: **while** $\alpha_s > 1$ **and** $b(s) - a(s) \le n\alpha_s$ **do** $\alpha_s := \alpha_s/2$ ;
$\ell$2: **if** $a(s) = b(s)$ **then** $V' := V' \setminus \{s\}$ ;
$\ell$3: **return** .

The correctness and the time complexity of the algorithm follow from the next lemmas.

**Lemma 3.3** COORDINATEWISE_SCALING *preserves conditions* (c1) *to* (c4).

**Proof.** As we mentioned above, conditions (c1) to (c4) are satisfied just after the execution of line $\ell$03. We note that Proposition 3.2 guarantees that $f_a^b$ is M-convex. The while loop, which is lines $\ell$04 to $\ell$12, consists of three cases.

The first case at line $\ell$08 implies that there exists $x_* \in \arg\min f$ with $x_*(v) = x(v)$ by condition (c3), Proposition 2.2 and Theorem 2.4 for $f_a^b$. Trivially, conditions (c1) to (c4) are satisfied after the execution of line $\ell$09.

We next consider the second case at line $\ell$10. By (c3) and (b) of Theorem 2.4 for $f_a^b$, there exists $x_1 \in \arg\min f_a^b \subseteq \arg\min f$ with $x_1(u_1) \le x(u_1) - 1$. Let us consider function CASE2, in which we have either $u \in W_u$ or $u \notin W_u$. Assume first that $u \in W_u$. By (c3) and (a) of Theorem 2.5 for $f_a^b$, there exists $x_2 \in \arg\min f$ with $a(u) \le x_2(u)$, for the updated $a$. The updated $a$ and $b$ have $a(u) \le b(u)$. By Proposition 2.2, there exists $x_* \in \arg\min f$ with $a(u) \le x_*(u) \le b(u)$. Thus, (c1) to (c4) are satisfied. Assume next that $u \notin W_u$. By (c3) and (a) of Theorem 2.6 for $f_a^b$, there exists $x_2 \in \arg\min f$ with $a(w) \le x_2(w)$, for the updated $a$. By Proposition 2.2, there exists $x_* \in \arg\min f$ with $a(w) \le x_*(w)$ and $x_*(u) \le b(u)$ for the updated $a$ and $b$, and hence, (c3) holds. Obviously, the updated $a$, $b$ and $x$ violate no other conditions at the end of line $\ell 6$. At lines $\ell 7$ and $\ell 8$, UPDATE_FACTOR reduces $\alpha_w$ and $\alpha_u$ respectively, preserving (c1) to (c4).

Similarly, the third case at $\ell 11$ also preserves (c1) to (c4). ∎

**Lemma 3.4** *The while loop in lines $\ell 04$ to $\ell 12$ of* COORDINATEWISE_SCALING *terminates in $O(n^2 \log(L/n))$ iterations.*

**Outline of Proof.** We divide the iterations of the while loop into three cases.
CLAIM A: The case at line $\ell 08$ occurs at most $n-1$ times.
[Proof of Claim A] Every time this case occurs, the set $V'$ is reduced by one element.

CLAIM B: The case of $u \in W_u$ in CASE2 or CASE3 occurs $O(n \log(L/n))$ times.
[Proof of Claim B] In this case, $\alpha_u$ must be updated at line $\ell 8$ of CASE2 or CASE3 at least once because $b(u) - a(u) < n\alpha_u$. Thus, this case occurs at most $\lceil \log(L/n) \rceil$ times for a fixed $u \in V$.

CLAIM C: The case of $u \notin W_u$ in CASE2 or CASE3 occurs $O(n^2 \log(L/n))$ times.

By Claims A, B and C, the while loop terminates in $O(n^2 \log(L/n))$ iterations. In the following, we prove Claim C.

[Outline of Proof of Claim C] To prove Claim C, we adopt two auxiliary vectors $a', b' \in \mathbf{Z}^V$ such that

$$b'(s) - x(s) \text{ and } x(s) - a'(s) \text{ are divisible by } \alpha_s, \tag{1}$$

$$b'(s) - a'(s) + 2\alpha_s \geq \alpha_s \lfloor (b(s) - a(s))/\alpha_s \rfloor, \tag{2}$$

$$a'(s) - 2\alpha_s < a(s) \leq a'(s) \leq x(s) \leq b'(s) \leq b(s) < b'(s) + 2\alpha_s \tag{3}$$

for any $s \in V$. We evaluate the number of the occurrences of the case in Claim C by using an integral number $\psi$ defined by

$$\psi = \sum_{s \in V} \frac{b'(s) - a'(s)}{\alpha_s}. \tag{4}$$

By (c2) and (3), $(b'(s) - a'(s))/\alpha_s \leq 2n$ holds. We can show that $\psi$ does not increase if no scaling factors are updated, and that either $\psi$ or a certain scaling factor is strictly reduced except in one case, which does not occur consecutively in a sense. ∎

By Lemmas 3.1, 3.3 and 3.4, our main result is obtained.

**Theorem 3.5** *Suppose that $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is an M-convex function with a bounded effective domain and that $x^0 \in \operatorname{dom} f$ is given. Algorithm* COORDINATEWISE_SCALING *finds a minimizer of $f$ in $O(Fn^3 \log(L/n))$ time, where $F$ is an upper bound of time to evaluate $f$ and $L = \max\{|x(u) - y(u)| \mid u \in V,\ x, y \in \operatorname{dom} f\}$.*

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