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完全対称有向グラフの均衡的 (C_4, C_6)-Bowtie 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_6 をそれぞれ 4 点、6 点を通る有向サイクルとする。1 点を共有する辺素な 2 個の有向サイクル C_4、C_6 からなるグラフを (C_4, C_6)-bowtie という。本研究では、完全対称有向グラフ K_n^* を (C_4, C_6)-bowtie 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 (C_4, C_6)-bowtie 分解; 完全対称有向グラフ; グラフ理論

Balanced (C_4, C_6)-Bowtie Decomposition Algorithm of Symmetric Complete Digraphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_6)-bowtie decomposition algorithm of the symmetric complete digraph K_n^*.

Keywords: Balanced (C_4, C_6)-bowtie decomposition; Symmetric complete digraph; Graph theory

1. Introduction

Let K_n^* denote *the symmetric complete digraph* of n vertices. Let C_4 and C_6 be *the directed* 4-*cycle* and *the directed* 6-*cycle*, respectively. *The* (C_4, C_6)-*bowtie* is a graph of edge-disjoint C_4 and C_6 with a common vertex and the common vertex is called *the center of the* (C_4, C_6)-*bowtie.*

When K_n^* is decomposed into edge-disjoint sum of (C_4, C_6)-bowties, we say that K_n^* *has a* (C_4, C_6)-*bowtie decomposition.* Moreover, when every vertex of K_n^* appears in the same number of (C_4, C_6)-bowties, we say that K_n^* *has a balanced* (C_4, C_6)-*bowtie decomposition* and this number is called *the replication number.*
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[3, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced (C_4, C_6)-bowtie decomposition of K_n^* is $n \equiv 1$ (mod 10).

2. Balanced (C_4, C_6)-bowtie decomposition of K_n^*

We use the following notation for a (C_4, C_6)-bowtie.

Notation. We denote a (C_4, C_6)-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9)\}$.

We have the following theorem.

Theorem. K_n^* has a balanced (C_4, C_6)-bowtie decomposition if and only if $n \equiv 1$ (mod 10).

Proof. (Necessity) Suppose that K_n^* has a balanced (C_4, C_6)-bowtie decomposition. Let b be the number of (C_4, C_6)-bowties and r be the replication number. Then $b = n(n-1)/10$ and $r = 9(n-1)/10$. Among r (C_4, C_6)-bowties having a vertex v of K_n^*, let r_1 and r_2 be the numbers of (C_4, C_6)-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2r_1 + r_2 = n - 1$. From these relations, $r_1 = (n-1)/10$ and $r_2 = 4(n-1)/5$. Therefore, $n \equiv 1$ (mod 10) is necessary.

(Sufficiency) We consider 2 cases.

Case 1. $n = 11$. Construct 11 (C_4, C_6)-bowties as follows:
$B_1 = \{(1,3,6,5), (1,2,7,4,10,8)\}$
$B_2 = \{(2,4,7,6), (2,3,8,5,11,9)\}$
$B_3 = \{(3,5,8,7), (3,4,9,6,1,10)\}$
$B_4 = \{(4,6,9,8), (4,5,10,7,2,11)\}$
$B_5 = \{(5,7,10,9), (5,6,11,8,3,1)\}$
$B_6 = \{(6,8,11,10), (6,7,1,9,4,2)\}$
$B_7 = \{(7,9,1,11), (7,8,2,10,5,3)\}$
$B_8 = \{(8,10,2,1), (8,9,3,11,6,4)\}$
$B_9 = \{(9,11,3,2), (9,10,4,1,7,5)\}$
$B_{10} = \{(10,1,4,3), (10,11,5,2,8,6)\}$
$B_{11} = \{(11,2,5,4), (11,1,6,3,9,7)\}$.
This decomposition can be written as follows:
$B_i = \{(i, i+2, i+5, i+4), (i, i+1, i+6, i+3, i+9, i+7)\}$ $(i = 1, 2, ..., 11)$,
where the additions $i + x$ are taken modulo 11 with residues $1, 2, ..., 11$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{11}^*.

Note. We consider the vertex set V of K_n^* as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Case 2. $n = 10t + 1$ **and** $t \geq 2$. Construct tn (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+t+1, i+1, i+3t+1), (i, i+2, i+4t+4, i+2t+2, i+7t+4, i+6t+2)\}$
$B_i^{(2)} = \{(i, i+t+2, i+3, i+3t+2), (i, i+1, i+4t+2, i+2t+1, i+7t+2, i+6t+1)\}$
$B_i^{(3)} = \{(i, i+t+3, i+5, i+3t+3), (i, i+3, i+4t+6, i+2t+3, i+7t+6, i+6t+3)\}$
$B_i^{(4)} = \{(i, i+t+4, i+7, i+3t+4), (i, i+4, i+4t+8, i+2t+4, i+7t+8, i+6t+4)\}$
...
$B_i^{(t-1)} = \{(i, i+2t-1, i+2t-3, i+4t-1), (i, i+t-1, i+6t-2, i+3t-1, i+9t-2, i+7t-1)\}$
$B_i^{(t)} = \{(i, i+2t, i+2t-1, i+4t), (i, i+t, i+6t, i+3t, i+9t, i+7t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_n^*.
This completes the proof.

Example 1. A balanced (C_4, C_6)-bowtie decomposition of K_{21}^*.
Construct 42 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+3, i+1, i+7), (i, i+2, i+12, i+6, i+18, i+14)\}$
$B_i^{(2)} = \{(i, i+4, i+3, i+8), (i, i+1, i+10, i+5, i+16, i+13)\}$
$(i = 1, 2, ..., 21)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{21}^*.

Example 2. A balanced (C_4, C_6)-bowtie decomposition of K_{31}^*.
Construct 93 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+4, i+1, i+10), (i, i+2, i+16, i+8, i+25, i+20)\}$
$B_i^{(2)} = \{(i, i+5, i+3, i+11), (i, i+1, i+14, i+7, i+23, i+19)\}$
$B_i^{(3)} = \{(i, i+6, i+5, i+12), (i, i+3, i+18, i+9, i+27, i+21)\}$
$(i = 1, 2, ..., 31)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{31}^*.

Example 3. A balanced (C_4, C_6)-bowtie decomposition of K_{41}^*.
Construct 164 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+5, i+1, i+13), (i, i+2, i+20, i+10, i+32, i+26)\}$
$B_i^{(2)} = \{(i, i+6, i+3, i+14), (i, i+1, i+18, i+9, i+30, i+25)\}$
$B_i^{(3)} = \{(i, i+7, i+5, i+15), (i, i+3, i+22, i+11, i+34, i+27)\}$
$B_i^{(4)} = \{(i, i+8, i+7, i+16), (i, i+4, i+24, i+12, i+36, i+28)\}$
$(i = 1, 2, ..., 41)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{41}^*.

Example 4. A balanced (C_4, C_6)-bowtie decomposition of K_{51}^*.
Construct 255 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+6, i+1, i+16), (i, i+2, i+24, i+12, i+39, i+32)\}$
$B_i^{(2)} = \{(i, i+7, i+3, i+17), (i, i+1, i+22, i+11, i+37, i+31)\}$
$B_i^{(3)} = \{(i, i+8, i+5, i+18), (i, i+3, i+26, i+13, i+41, i+33)\}$
$B_i^{(4)} = \{(i, i+9, i+7, i+19), (i, i+4, i+28, i+14, i+43, i+34)\}$
$B_i^{(5)} = \{(i, i+10, i+9, i+20), (i, i+5, i+30, i+15, i+45, i+35)\}$
$(i = 1, 2, ..., 51)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{51}^*.

References

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