完全グラフの均衡的 (C_4, C_6)-Bowtie 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_6 をそれぞれ 4 点、6 点を通るサイクルとする。1 点を共有する辺素な 2 個のサイクル C_4、C_6 からなるグラフを (C_4, C_6)-bowtie という。本研究では、完全グラフ K_n を (C_4, C_6)-bowtie 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード : 均衡的 (C_4, C_6)-bowtie 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_6)-Bowtie Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_6)-bowtie decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_6)-bowtie decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_6 be *the* 4*-cycle* and *the* 6*-cycle*, respectively. *The* (C_4, C_6)*-bowtie* is a graph of edge-disjoint C_4 and C_6 with a common vertex and the common vertex is called *the center of the* (C_4, C_6)*-bowtie.*
When K_n is decomposed into edge-disjoint sum of (C_4, C_6)-bowties, we say that K_n *has a* (C_4, C_6)*-bowtie decomposition.* Moreover, when every vertex of K_n appears in the same number of (C_4, C_6)-bowties, we say that K_n *has a balanced* (C_4, C_6)*-bowtie decomposition* and this number is called

the replication number.
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[3, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a bowtie system.*
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced (C_4, C_6)-bowtie decomposition of K_n is $n \equiv 1 \pmod{20}$.

2. Balanced (C_4, C_6)-bowtie decomposition of K_n

We use the following notation for a (C_4, C_6)-bowtie.

Notation. We denote a (C_4, C_5)-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9)\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_6)-bowtie decomposition if and only if $n \equiv 1 \pmod{20}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_6)-bowtie decomposition. Let b be the number of (C_4, C_6)-bowties and r be the replication number. Then $b = n(n-1)/20$ and $r = 9(n-1)/20$. Among r (C_4, C_6)-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_6)-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/20$ and $r_2 = 2(n-1)/5$. Therefore, $n \equiv 1 \pmod{20}$ is necessary.

(Sufficiency) We consider 2 cases.

Case 1. $n = 21$. Construct 21 (C_4, C_6)-bowties as follows:
$B_1 = \{(1,4,10,5), (1,2,9,20,11,3)\}$
$B_2 = \{(2,5,11,6), (2,3,10,21,12,4)\}$
$B_3 = \{(3,6,12,7), (3,4,11,1,13,5)\}$
$B_4 = \{(4,7,13,8), (4,5,12,2,14,6)\}$
$B_5 = \{(5,8,14,9), (5,6,13,3,15,7)\}$
$B_6 = \{(6,9,15,10), (6,7,14,4,16,8)\}$
$B_7 = \{(7,10,16,11), (7,8,15,5,17,9)\}$
$B_8 = \{(8,11,17,12), (8,9,16,6,18,10)\}$
$B_9 = \{(9,12,18,13), (9,10,17,7,19,11)\}$
$B_{10} = \{(10,13,19,14), (10,11,18,8,20,12)\}$
$B_{11} = \{(11,14,20,15), (11,12,19,9,21,13)\}$
$B_{12} = \{(12,15,21,16), (12,13,20,10,1,14)\}$
$B_{13} = \{(13,16,1,17), (13,14,21,11,2,15)\}$
$B_{14} = \{(14,17,2,18), (14,15,1,12,3,16)\}$
$B_{15} = \{(15,18,3,19), (15,16,2,13,4,17)\}$
$B_{16} = \{(16,19,4,20), (16,17,3,14,5,18)\}$
$B_{17} = \{(17,20,5,21), (17,18,4,15,6,19)\}$
$B_{18} = \{(18,21,6,1), (18,19,5,16,7,20)\}$
$B_{19} = \{(19,1,7,2), (19,20,6,17,8,21)\}$
$B_{20} = \{(20,2,8,3), (20,21,7,18,9,1)\}$
$B_{21} = \{(21,3,9,4), (21,1,8,19,10,2)\}$.

This decomposition can be written as follows:
$B_i = \{(i, i+3, i+9, i+4), (i, i+1, i+8, i+19, i+10, i+2)\}$ $(i = 1, 2, ..., 21)$,
where the additions $i + x$ are taken modulo 19 with residues $1, 2, ..., 21$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{21}.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Case 2. $n = 20t + 1$ **and** $t \geq 2$. Construct tn (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+2t+1, i+17t+2, i+3t+1), (i, i+1, i+6t+2, i+16t+3, i+8t+2, i+t+1)\}$
$B_i^{(2)} = \{(i, i+2t+2, i+17t+4, i+3t+2), (i, i+2, i+6t+4, i+16t+6, i+8t+4, i+t+2)\}$
$B_i^{(3)} = \{(i, i+2t+3, i+17t+6, i+3t+3), (i, i+3, i+6t+6, i+16t+9, i+8t+6, i+t+3)\}$
...
$B_i^{(t-2)} = \{(i, i+3t-2, i+19t-4, i+4t-2), (i, i+t-2, i+8t-4, i+19t-6, i+10t-4, i+2t-2)\}$
$B_i^{(t-1)} = \{(i, i+3t-1, i+19t-2, i+4t-1), (i, i+t, i+8t, i+19t, i+10t, i+2t)\}$
$B_i^{(t)} = \{(i, i+3t, i+19t, i+4t), (i, i+t-1, i+8t-2, i+19t-3, i+10t-2, i+2t-1)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_n.
This completes the proof.

Example 1. A balanced (C_4, C_6)-bowtie decomposition of K_{41}.
Construct 82 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+5, i+36, i+7), (i, i+2, i+16, i+38, i+20, i+4)\}$
$B_i^{(2)} = \{(i, i+6, i+38, i+8), (i, i+1, i+14, i+35, i+18, i+3)\}$
$(i = 1, 2, ..., 41)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{41}.

Example 2. A balanced (C_4, C_6)-bowtie decomposition of K_{61}.
Construct 183 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+7, i+53, i+10), (i, i+1, i+20, i+51, i+26, i+4)\}$
$B_i^{(2)} = \{(i, i+8, i+55, i+11), (i, i+3, i+24, i+57, i+30, i+6)\}$
$B_i^{(3)} = \{(i, i+9, i+57, i+12), (i, i+2, i+22, i+54, i+28, i+5)\}$
$(i = 1, 2, ..., 61)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{61}.

Example 3. A balanced (C_4, C_6)-bowtie decomposition of K_{81}.
Construct 324 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+9, i+70, i+13), (i, i+1, i+26, i+67, i+34, i+5)\}$
$B_i^{(2)} = \{(i, i+10, i+72, i+14), (i, i+2, i+28, i+70, i+36, i+6)\}$
$B_i^{(3)} = \{(i, i+11, i+74, i+15), (i, i+4, i+32, i+76, i+40, i+8)\}$
$B_i^{(4)} = \{(i, i+12, i+76, i+16), (i, i+3, i+30, i+73, i+38, i+7)\}$
$(i = 1, 2, ..., 81)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{81}.

Example 4. A balanced (C_4, C_6)-bowtie decomposition of K_{101}.
Construct 505 (C_4, C_6)-bowties as follows:
$B_i^{(1)} = \{(i, i+11, i+87, i+16), (i, i+1, i+32, i+83, i+42, i+6)\}$
$B_i^{(2)} = \{(i, i+12, i+89, i+17), (i, i+2, i+34, i+86, i+44, i+7)\}$

$B_i^{(3)} = \{(i, i+13, i+91, i+18), (i, i+3, i+36, i+89, i+46, i+8)\}$
$B_i^{(4)} = \{(i, i+14, i+93, i+19), (i, i+5, i+40, i+95, i+50, i+10)\}$
$B_i^{(5)} = \{(i, i+15, i+95, i+20), (i, i+4, i+38, i+92, i+48, i+9)\}$
$(i = 1, 2, ..., 101)$.
Then they comprise a balanced (C_4, C_6)-bowtie decomposition of K_{101}.

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