完全グラフの均衡的 (C_4, C_9, C_9)-Trefoil 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_9 をそれぞれ 4 点、9 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_9、C_9 からなるグラフを (C_4, C_9, C_9)-trefoil という。本研究では、完全グラフ K_n を (C_4, C_9, C_9)-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 (C_4, C_9, C_9)-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_9, C_9)-Trefoil Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_9, C_9)-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_9, C_9)-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_9 be *the* 4-*cycle* and *the* 9-*cycle*, respectively. *The* (C_4, C_9, C_9)-*trefoil* is a graph of 3 edge-disjoint cycles C_4, C_9 and C_9 with a common vertex and the common vertex is called *the center of the* (C_4, C_9, C_9)-*trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_9, C_9)-trefoils, we say that K_n *has a* (C_4, C_9, C_9)-*trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_9, C_9)-trefoils, we say that K_n *has a balanced* (C_4, C_9, C_9)-*trefoil decomposition* and

this number is called *the replication number.*

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_9, C_9)-trefoil decomposition of K_n is to be known as *a balanced (C_4, C_9, C_9)-trefoil system.*

2. Balanced (C_4, C_9, C_9)-trefoil decomposition of K_n

We use the following notation for a (C_4, C_9, C_9)-trefoil.

Notation. We denote a (C_4, C_9, C_9)-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1$, $v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_1$, $v_1 - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_{20} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}), (v_1, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_9, C_9)-trefoil decomposition if and only if $n \equiv 1 \pmod{44}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_9, C_9)-trefoil decomposition. Let b be the number of (C_4, C_9, C_9)-trefoils and r be the replication number. Then $b = n(n-1)/44$ and $r = 20(n-1)/44$. Among r (C_4, C_9, C_9)-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_9, C_9)-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/44$ and $r_2 = 19(n-1)/44$. Therefore, $n \equiv 1 \pmod{44}$ is necessary.

(Sufficiency) Put $n = 44t + 1$. We consider 3 cases.

Case 1. $t = 1$, $n = 45$. (Example 1.) Construct a balanced (C_4, C_9, C_9)-trefoil decomposition of K_{45} as follows:
$B_1 = \{(1,12,26,14), (1,2,17,20,25,35,7,28,9), (1,3,19,23,29,38,11,31,8)\}$
$B_2 = \{(2,13,27,15), (2,3,18,21,26,36,8,29,10), (2,4,20,24,30,39,12,32,9)\}$
$B_3 = \{(3,14,28,16), (3,4,19,22,27,37,9,30,11), (3,5,21,25,31,40,13,33,10)\}$
$B_4 = \{(4,15,29,17), (4,5,20,23,28,38,10,31,12), (4,6,22,26,32,41,14,34,11)\}$
$B_5 = \{(5,16,30,18), (5,6,21,24,29,39,11,32,13), (5,7,23,27,33,42,15,35,12)\}$
$B_6 = \{(6,17,31,19), (6,7,22,25,30,40,12,33,14), (6,8,24,28,34,43,16,36,13)\}$
$B_7 = \{(7,18,32,20), (7,8,23,26,31,41,13,34,15), (7,9,25,29,35,44,17,37,14)\}$
$B_8 = \{(8,19,33,21), (8,9,24,27,32,42,14,35,16), (8,10,26,30,36,45,18,38,15)\}$
$B_9 = \{(9,20,34,22), (9,10,25,28,33,43,15,36,17), (9,11,27,31,37,1,19,39,16)\}$
$B_{10} = \{(10,21,35,23), (10,11,26,29,34,44,16,37,18), (10,12,28,32,38,2,20,40,17)\}$
$B_{11} = \{(11,22,36,24), (11,12,27,30,35,45,17,38,19), (11,13,29,33,39,3,21,41,18)\}$
$B_{12} = \{(12,23,37,25), (12,13,28,31,36,1,18,39,20), (12,14,30,34,40,4,22,42,19)\}$
$B_{13} = \{(13,24,38,26), (13,14,29,32,37,2,19,40,21), (13,15,31,35,41,5,23,43,20)\}$
$B_{14} = \{(14,25,39,27), (14,15,30,33,38,3,20,41,22), (14,16,32,36,42,6,24,44,21)\}$
$B_{15} = \{(15,26,40,28), (15,16,31,34,39,4,21,42,23), (15,17,33,37,43,7,25,45,22)\}$
$B_{16} = \{(16,27,41,29), (16,17,32,35,40,5,22,43,24), (16,18,34,38,44,8,26,1,23)\}$
$B_{17} = \{(17,28,42,30), (17,18,33,36,41,6,23,44,25), (17,19,35,39,45,9,27,2,24)\}$
$B_{18} = \{(18,29,43,31), (18,19,34,37,42,7,24,45,26), (18,20,36,40,1,10,28,3,25)\}$

$B_{19} = \{(19,30,44,32),(19,20,35,38,43,8,25,1,27),(19,21,37,41,2,11,29,4,26)\}$
$B_{20} = \{(20,31,45,33),(20,21,36,39,44,9,26,2,28),(20,22,38,42,3,12,30,5,27)\}$
$B_{21} = \{(21,32,1,34),(21,22,37,40,45,10,27,3,29),(21,23,39,43,4,13,31,6,28)\}$
$B_{22} = \{(22,33,2,35),(22,23,38,41,1,11,28,4,30),(22,24,40,44,5,14,32,7,29)\}$
$B_{23} = \{(23,34,3,36),(23,24,39,42,2,12,29,5,31),(23,25,41,45,6,15,33,8,30)\}$
$B_{24} = \{(24,35,4,37),(24,25,40,43,3,13,30,6,32),(24,26,42,1,7,16,34,9,31)\}$
$B_{25} = \{(25,36,5,38),(25,26,41,44,4,14,31,7,33),(25,27,43,2,8,17,35,10,32)\}$
$B_{26} = \{(26,37,6,39),(26,27,42,45,5,15,32,8,34),(26,28,44,3,9,18,36,11,33)\}$
$B_{27} = \{(27,38,7,40),(27,28,43,1,6,16,33,9,35),(27,29,45,4,10,19,37,12,34)\}$
$B_{28} = \{(28,39,8,41),(28,29,44,2,7,17,34,10,36),(28,30,1,5,11,20,38,13,35)\}$
$B_{29} = \{(29,40,9,42),(29,30,45,3,8,18,35,11,37),(29,31,2,6,12,21,39,14,36)\}$
$B_{30} = \{(30,41,10,43),(30,31,1,4,9,19,36,12,38),(30,32,3,7,13,22,40,15,37)\}$
$B_{31} = \{(31,42,11,44),(31,32,2,5,10,20,37,13,39),(31,33,4,8,14,23,41,16,38)\}$
$B_{32} = \{(32,43,12,45),(32,33,3,6,11,21,38,14,40),(32,34,5,9,15,24,42,17,39)\}$
$B_{33} = \{(33,44,13,1),(33,34,4,7,12,22,39,15,41),(33,35,6,10,16,25,43,18,40)\}$
$B_{34} = \{(34,45,14,2),(34,35,5,8,13,23,40,16,42),(34,36,7,11,17,26,44,19,41)\}$
$B_{35} = \{(35,1,15,3),(35,36,6,9,14,24,41,17,43),(35,37,8,12,18,27,45,20,42)\}$
$B_{36} = \{(36,2,16,4),(36,37,7,10,15,25,42,18,44),(36,38,9,13,19,28,1,21,43)\}$
$B_{37} = \{(37,3,17,5),(37,38,8,11,16,26,43,19,45),(37,39,10,14,20,29,2,22,44)\}$
$B_{38} = \{(38,4,18,6),(38,39,9,12,17,27,44,20,1),(38,40,11,15,21,30,3,23,45)\}$
$B_{39} = \{(39,5,19,7),(39,40,10,13,18,28,45,21,2),(39,41,12,16,22,31,4,23,1)\}$
$B_{40} = \{(40,6,20,8),(40,41,11,14,19,29,1,22,3),(40,42,13,17,23,32,5,25,2)\}$
$B_{41} = \{(41,7,21,9),(41,42,12,15,20,30,2,23,4),(41,43,14,18,24,33,6,26,3)\}$
$B_{42} = \{(42,8,22,10),(42,43,13,16,21,31,3,24,5),(42,44,15,19,25,34,7,27,4)\}$
$B_{43} = \{(43,9,23,11),(43,44,14,17,22,32,4,25,6),(43,45,16,20,26,35,8,28,5)\}$
$B_{44} = \{(44,10,24,12),(44,45,15,18,23,33,5,26,7),(44,1,17,21,27,36,9,29,6)\}$
$B_{45} = \{(45,11,25,13),(45,1,16,19,24,34,6,27,8),(45,2,18,22,28,37,10,30,7)\}.$

This decomposition can be written as follows:
$B_i = \{(i,i+11,i+25,i+13),(i,i+1,i+16,i+19,i+24,i+34,i+6,i+27,i+8),(i,i+2,i+18,i+22,i+28,i+37,i+10,i+30,i+7)\}$ $(i=1,2,...,45)$,
where the additions $i+x$ are taken modulo 45 with residues $1,2,...,45$.

Note. We consider the vertex set V of K_n as $V=\{1,2,...,n\}$.
The additions $i+x$ are taken modulo n with residues $1,2,...,n$.

Case 2. $t=2$, $n=89$. **(Example 2.)** Construct a balanced (C_4,C_9,C_9)-trefoil decomposition of K_{89} as follows:
$B_i^{(1)} = \{(i,i+27,i+48,i+25),(i,i+1,i+30,i+38,i+47,i+64,i+8,i+60,i+16),(i,i+2,i+32,i+39,i+49,i+67,i+12,i+63,i+15)\}$
$B_i^{(2)} = \{(i,i+22,i+50,i+26),(i,i+3,i+34,i+40,i+51,i+70,i+21,i+56,i+14),(i,i+4,i+36,i+41,i+53,i+73,i+20,i+59,i+13)\}$
$(i=1,2,...,89)$.

Case 3. $t \geq 3$, $n=44t+1$.
First, consider a sequence $S: g_1,g_2,g_3,...,g_{2t}$ for 2 subcases.
Subcase 3.1. $t \equiv 1 \pmod 2$, $t \geq 3$. Put $t=2p+1$.
Consider a sequence $S: g_1,g_2,g_3,...,g_{4p+2}$ with
$S_1: g_1,g_2,g_3,...,g_{2p-2}$,
$S_2: g_{2p-1},g_{2p},g_{2p+1},g_{2p+2}$,

$S_3: g_{2p+3}, g_{2p+5}, g_{2p+7}, ..., g_{4p+1}$,
$S_4: g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+2}$
such as
$S_1: 24t+4, 24t+7, 24t+10, ..., 24t+6p-5$
$S_2: 24t+6p-1, 24t+6p+2, 30t+3, 24t+6p+8$
$S_3: 24t+6p+12, 24t+6p+18, 24t+6p+24, ..., 30t$
$S_4: 24t+6p+11, 24t+6p+17, 24t+6p+23, ..., 30t-1$.
Subcase 3.2. $t \equiv 0 \pmod 2$, $t \geq 4$. Put $t = 2p$.
Consider a sequence $S: g_1, g_2, g_3, ..., g_{4p}$ with
$S_1: g_1, g_2, g_3, ..., g_{2p-3}$,
$S_2: g_{2p-2}, g_{2p-1}, g_{2p}, g_{2p+1}$,
$S_3: g_{2p+2}, g_{2p+4}, g_{2p+6}, ..., g_{4p-2}$,
$S_4: g_{2p+3}, g_{2p+5}, g_{2p+7}, ..., g_{4p-1}$,
$S_5: g_{4p}$
such as
$S_1: 24t+4, 24t+7, 24t+10, ..., 24t+6p-8$
$S_2: 24t+6p-4, 24t+6p-1, 30t+3, 24t+6p+5$
$S_3: 24t+6p+9, 24t+6p+15, 24t+6p+21, ..., 30t-3$
$S_4: 24t+6p+8, 24t+6p+14, 24t+6p+20, ..., 30t-4$
$S_5: 30t$.
Next, construct n (C_4, C_9, C_9)-$3t$-foils as follows:
$\{(i, i+31t+1, i+41t+2, i+11t+1),$
$(i, i+31t+2, i+41t+4, i+11t+2),$
$...,$
$(i, i+32t, i+43t, i+12t),$
$(i, i+1, i+6t+2, i+16t+2, i+32t+3, i+36t+3, i+6t+3, i+g_1, i+4t+1),$
$(i, i+2, i+6t+4, i+16t+3, i+32t+5, i+36t+4, i+6t+5, i+g_2, i+4t+2),$
$...,$
$(i, i+2t, i+10t, i+18t+1, i+36t+1, i+38t+2, i+10t+1, i+g_{2t}, i+6t)\}$
$(i = 1, 2, ..., n)$.
Last, decompose each (C_4, C_9, C_9)-$3t$-foil into t (C_4, C_9, C_9)-trefoils.
Then they comprise a balanced (C_4, C_9, C_9)-trefoil decomposition of K_n.

This completes the proof.

Note. *The* (C_4, C_9, C_9)-*$3t$-foil* is a graph of $3t$ edge-disjoint cycles of t C_4's and $2t$ C_9's with a common vertex.

Example 3. A balanced (C_4, C_9, C_9)-trefoil decomposition of K_{133}.
$B_i^{(1)} = \{(i, i+94, i+125, i+34), (i, i+1, i+20, i+50, i+99, i+111, i+21, i+77, i+13), (i, i+2, i+22, i+51, i+101, i+112, i+23, i+80, i+14)\}$
$B_i^{(2)} = \{(i, i+95, i+127, i+35), (i, i+3, i+24, i+52, i+103, i+113, i+25, i+93, i+15), (i, i+4, i+26, i+53, i+105, i+114, i+27, i+86, i+16)\}$
$B_i^{(3)} = \{(i, i+96, i+129, i+36), (i, i+5, i+28, i+54, i+107, i+115, i+29, i+90, i+17), (i, i+6, i+30, i+55, i+109, i+116, i+31, i+89, i+18)\}$
$(i = 1, 2, ..., 133)$.

Example 4. A balanced (C_4, C_9, C_9)-trefoil decomposition of K_{177}.
$B_i^{(1)} = \{(i, i+125, i+166, i+45), (i, i+1, i+26, i+66, i+131, i+147, i+27, i+100, i+17), (i, i+2, i+28, i+67, i+133, i+148, i+29, i+104, i+18)\}$

$B_i^{(2)} = \{(i, i+126, i+168, i+46), (i, i+3, i+30, i+68, i+135, i+149, i+31, i+107, i+19), (i, i+4, i+32, i+69, i+137, i+150, i+33, i+123, i+20)\}$
$B_i^{(3)} = \{(i, i+127, i+170, i+47), (i, i+5, i+34, i+70, i+139, i+151, i+35, i+113, i+21), (i, i+6, i+36, i+71, i+141, i+152, i+37, i+117, i+22)\}$
$B_i^{(4)} = \{(i, i+128, i+172, i+48), (i, i+7, i+38, i+72, i+143, i+153, i+39, i+116, i+23), (i, i+8, i+40, i+73, i+145, i+154, i+41, i+120, i+24)\}$
$(i = 1, 2, ..., 177)$.

Example 5. A balanced (C_4, C_9, C_9)-trefoil decomposition of K_{221}.
$B_i^{(1)} = \{(i, i+156, i+207, i+56), (i, i+1, i+32, i+82, i+163, i+183, i+33, i+124, i+21), (i, i+2, i+34, i+83, i+165, i+184, i+35, i+127, i+22)\}$
$B_i^{(2)} = \{(i, i+157, i+209, i+57), (i, i+3, i+36, i+84, i+167, i+185, i+37, i+131, i+23), (i, i+4, i+38, i+85, i+169, i+186, i+39, i+134, i+24)\}$
$B_i^{(3)} = \{(i, i+158, i+211, i+58), (i, i+5, i+40, i+86, i+171, i+187, i+41, i+153, i+25), (i, i+6, i+42, i+87, i+173, i+188, i+43, i+140, i+26)\}$
$B_i^{(4)} = \{(i, i+159, i+213, i+59), (i, i+7, i+44, i+88, i+175, i+189, i+45, i+144, i+27), (i, i+8, i+46, i+89, i+177, i+190, i+47, i+143, i+28)\}$
$B_i^{(5)} = \{(i, i+160, i+215, i+60), (i, i+9, i+48, i+90, i+179, i+191, i+49, i+150, i+29), (i, i+10, i+50, i+91, i+181, i+192, i+51, i+149, i+30)\}$
$(i = 1, 2, ..., 221)$.

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