A Simple Constant Time Enumeration Algorithm for Free Trees

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Abstract: In this paper, we consider a problem of enumerating all trees with n vertices where n is given as input. Any output tree have to be isomorphic to no other output tree. We propose an algorithm for solving this problem. The time completiy of the algorithm is exactly constant time per output tree, which means computation time between any consecutive two trees is constant time. In existing studies, many algorithms are proposed for solving this problem, however the time complexity of the best algorithm is average constant time, which may takes long time between two consecutive output trees.
Keywords: enumeration, generation, free tree

無順序根無し木を列挙するシンプルなアルゴリズム

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抄録: 本稿では, 頂点数が n の木を列挙する問題を考える。ただし、同型な木は１度しか出力しないものとする。この問題に対し、木１つあたりの計算時間が正確に定数時間であるアルゴリズムを提案する。正確に定数時間であるとは、任意の連続する２つの出力の間にかかる時間が定数時間であるということである。この問題に対して多くのアルゴリズムが既存研究で提案されているが、これらの計算時間は、最良のものでも木１つあたりの平均時間が定数時間であり、連続する２つの出力の間に時間がかかる可能性もある。
キーワード： 列挙, 生成, 根無し木

1 Introduction

Recently, enumeration problems are frequently appearing in the models of many scientific area, such as artificial intelligence, computational linguistics, data mining, and graph mining. In these areas, fundamental graph objects subset families, sequences of letters with several constraints are needed to be enumerated efficiently. These are done by modified versions of fundamental enumeration algorithms proposed in the area of discrete algorithms. To develop the researches of these areas, efficient enumeration algorithms for many objects are needed.

In this paper, we deal with enumeration of trees composed of n vertices without outputting two isomorphic trees. This problem is known as enumeration of free trees. For this kind of subgraph enumeration problems, many algorithms have been proposed [B80, LN01, LR99, N02, M98, R78, W86]. Algorithms for enumerating free trees are also already known. The best algorithm [W86] runs in time proportional to the number of trees. However, the time needed to generate each tree may not be bounded by a constant, even though it is "on average".

In this paper we give a simple algorithm to generate, without repetition, all trees with exactly n vertices and diameter d. By using this, we can solve enumeration of free trees. Our algorithm generates each tree in constant time. It does not output each tree entirely, but outputs the difference from the preceding tree.

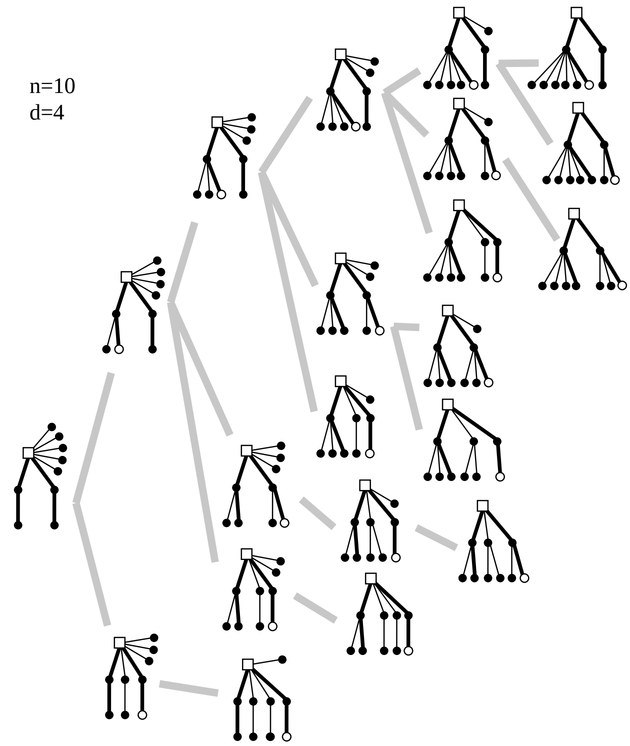

Figure 1: The family tree $\mathcal{T}_{10,4}$ for 10 vertices and diameter 4.

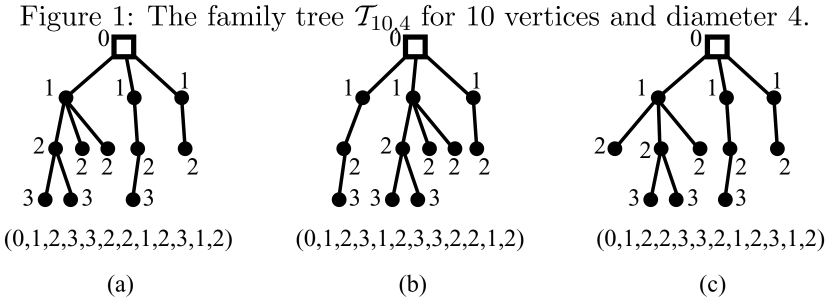

Figure 2: The depth sequences.

The main idea of our algorithm is to define an implicit tree connecting all the trees with n vertices. We call the tree "family tree" of trees (see Fig. 1). The algorithm traverses the tree and outputs trees corresponding to the current visiting vertex. Furthermore, the algorithm generates all trees so that each tree can be obtained from the preceding tree by at most three operations, where each operation consists of a deletion of a vertex and an addition of a vertex. Therefore the derived sequence of trees is a kind of combinatorial Gray code [J80, S97, W89] for trees with n vertices and diameter d. A Gray code [R00] is a cyclic sequence of all 2^k bitstrings of length k, such that each bitstring differs from the preceding one in a small number of bit entries.

2 Preliminaries

In this section we give some definitions.

Let G be a connected graph with n vertices. The *length* of a path is the number of edges in the path. The *diameter* of G is the maximum number of edges in a path connecting two vertices in G if G includes no cycle. A *rooted* tree is a tree with one vertex r chosen as its *root* . Un-rooted trees are called *free trees.* the *depth* of vertex v is the number of edges in the paths connecting v and r. An *ordered tree* is a rooted tree with left-to-right ordering specified for the children of each vertex. We denote by $T(v)$ the ordered subtree of an ordered tree T consisting of a vertex v and all descendants of v that preserve the left-to-right ordering for the children of each vertex.

Let T be an ordered tree with n vertices, and $(v_1, v_2, \cdots, v_n)$ be the sequence of the vertices of T in preorder [A95], which is given by the visiting order of a depth first search, searching the descendants of the children of any vertex in the left-to-right order. The sequence $L(T) = (dep(v_1), dep(v_2), \cdots, dep(v_n))$ is called the *depth sequence* of T. Some examples are shown in Fig. 2. Note that those trees in Fig. 2

are isomorphic as rooted trees, but non-isomorphic as ordered trees.

Let T_1 and T_2 be two ordered trees having n vertices, and $L(T_1) = (a_1, a_2, \cdots, a_n)$ and $L(T_2) = (b_1, b_2, \cdots, b_n)$ be their depth sequences. If $a_i = b_i$ for each $i = 1, 2, \cdots, j-1$ (possibly $j = 1$) and $a_j > b_j$, then we say that $L(T_1)$ is *heavier* than $L(T_2)$, and write $L(T_1) > L(T_2)$.

3 The Family Tree

In this section, we give the definition of family tree, which is the key idea of our algorithm. In Section 3 and 4, we only consider the case where the diameter is even.

If a tree has $n \geq 3$ vertices and diameter 2, then the number of such a tree is exactly one, which is $K_{1,n-1}$. In the rest of this section, we assume that the diameter is $2k \geq 4$.

Let T be a tree with the diameter $2k$. Let $v_0, v_1, \cdots, v_{2k}$ be a path in T having length $2k$. One can observe that T may have many such paths, but the vertex v_k, called *the center* of T, is unique [W01, p72]. Let R be the rooted tree obtained from T by setting v_k to the root. Let H be the ordered tree corresponding to R that has the heaviest depth sequence $L(H)$ among all the ordered tree corresponding to R. We say that H is the *left-heavy embedding* of T. For example, the ordered tree in Fig. 2(a) is the left-heavy embedding of a rooted tree, however the trees in Fig. 2(b) and (c) are not, since the one in Fig. 2(a) is heavier. Note that left-heavy embedding is uniquely defined for any tree with even diameter, and no two trees share the same left-heavy embedding.

Let $\mathcal{S}_{n,2k}$ be the set of all left-heavy embeddings of trees with exactly n vertices and diameter $2k$. If we generate all ordered trees in $\mathcal{S}_{n,2k}$, then it also means the generation of all trees with exactly n vertices and diameter $2k$. Hence, we are going to generate all ordered trees in $\mathcal{S}_{n,2k}$.

To characterize the ordered trees of $\mathcal{S}_{n,2k}$, we have the following lemma.

Lemma 1 *An ordered tree H is the left-heavy embedding of a rooted tree if and only if for every pair of consecutive child vertices v_1 and v_2, that appear in this order in the left-to-right ordering, $L(T(v_1)) \geq L(T(v_2))$ holds.*

Proof : By contradiction. ∎

In the rest of the paper the condition "$L(T(v_1)) \geq L(T(v_2))$ for each consecutive child vertices v_1 and v_2", is called *the left-heavy condition.*

Let H be a left-heavy embedding in $\mathcal{S}_{n,2k}$ rooted at r_k. Let $c_1, c_2, \cdots, c_k$ be the children of r_k. Assume they appear in this order in the left-to-right ordering. We say that c_i, $3 \leq i \leq k$ is a *waiting vertex* if $c_i, c_{i+1}, \cdots, c_k$ are leaves. Since H has a path of length $2k$ with the center r_k, one can observe that c_1 and c_2 have a descendant at depth k, respectively. Thus, neither c_1 nor c_2 are leaves. We denote by $A(H)$ the ordered tree derived from H by removing all (possibly none) waiting vertices. We say that $A(H)$ is *the active tree* of H. Note that the diameter of $A(H)$ is also $2k$.

Let c_a be the rightmost child of the root r_k in $A(H)$. Let $P_{right} = (v_0 = r_k, v_1 = c_a, v_2, \cdots, v_x)$ be the path in $A(H)$ such that v_i is the rightmost child of v_{i-1} for each i, $1 \leq i \leq x$, and v_x is a leaf in $A(H)$. We call P_{right} *the right path* of H. If $v_1 = c_2$ and $H(v_1)$ is a path, then we say H is *right empty.* Note that $H(v_1)$ is the ordered subtree of H induced by v_1 and all descendants of v_1. Similarly, let $P_{left} = (u_0 = r_k, u_1 = c_1, u_2, \cdots, u_y)$ be the path in $A(H)$ such that u_1 is the leftmost child of u_0, and u_i is the rightmost child of u_{i-1} for each i, $2 \leq i \leq y$, and u_y is a leaf in $A(H)$. We call P_{left} *the left path* of H. If $H(u_1)$ is a path, then we say H is *left empty.* The right and left paths are depicted as thick lines in Fig. 1.

If H is not right empty then v_x is called *the active leaf* of H. Otherwise, if H is not left empty then u_y is called *the active leaf* of H. Otherwise, $A(H)$ is a path of length $2k$, and H has no active leaf.

Assume that H is an ordered tree in $\mathcal{S}_{n,2k}$ that has an active leaf. We denote by $P(H)$ the ordered tree derived from H by (i) removing the active leaf of H, then (ii) adding one leaf as the rightmost child of the root. We say that $P(H)$ is *the parent tree* of H and H is *a child tree* of $P(H)$. We have the following lemma.

Lemma 2 *For any ordered tree H in $\mathcal{S}_{n,2k}$ having an active leaf, $P(H) \in \mathcal{S}_{n,2k}$. Moreover, H is heavier than $P(H)$.*

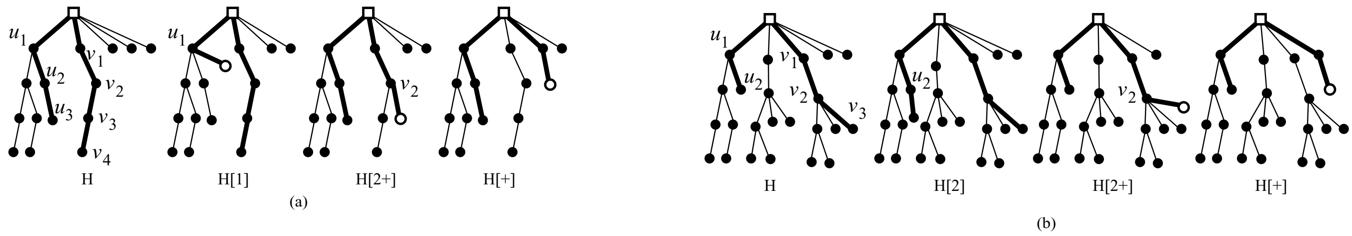

Figure 3: The possible child trees.

Proof : Removing the active leaf and then adding one leaf as the rightmost child of the root never destroys the left-heavy condition. And the number of vertices in the derived tree is still n. Furthermore the diameter of the derived tree is again $2k$. Thus any derived tree is also in $\mathcal{S}_{n,2k}$.

The proof for the second half of the claim is omitted. ∎

From the lemma, we can see that the parent-child relationship is acyclic, and its graph representation forms a tree whose root corresponds to the ordered tree with no active leaf. we call the tree *the family tree* of $\mathcal{S}_{n,2k}$, denoted by $\mathcal{T}_{n,2k}$. For instance, $\mathcal{T}_{10,4}$ is shown in Fig. 1.

4 Algorithm

In this section we give an algorithm to traverse the family tree.

If we can generate all child trees of a given ordered tree of $\mathcal{S}_{n,2k}$, then in a recursive manner we can traverse the family tree. This means we can generate all trees with exactly n vertices and diameter $2k$. Now we are going to generate all child trees of a given ordered tree. Let H be an ordered tree in $\mathcal{S}_{n,2k}$. Let $P_{right} = (v_0 = r_k, v_1, \cdots, v_x)$ be the right path of H, and $P_{left} = (u_0 = r_k, u_1, \cdots, u_y)$ be the left path of H. We construct some ordered trees by slightly modifying H as follows. Set $x^{'} = \min\{x, k-1\}$ and $y^{'} = \min\{y, k-1\}$. If H has at least one waiting vertex and H is right empty, then we define $H[i]$, $1 \le i \le y^{'}$, as the ordered tree derived from H by (i) removing the rightmost waiting vertex, then (ii) adding a new vertex as the rightmost child of u_i. See Fig. 3 for some examples. Note that the constraint $i \le y^{'} \le k-1$ ensures that the diameter of $H[i]$ remains $2k$. If H has at least one waiting vertex, then we define $H[i+]$, $1 \le i \le x^{'}$, as the ordered tree derived from H by (i) removing the rightmost waiting vertex, then (ii) adding a new vertex as the rightmost child of v_i. See some examples in Fig. 3. If H has at least two waiting vertices, then we define $H[+]$ as the ordered tree derived from H by (i) removing the rightmost waiting vertex, then (ii) adding a new vertex as the only child vertex of the leftmost waiting vertex. See Fig. 3.

We can observe that each child tree of H is in $\{H[1], H[2], \cdots, H[y^{'}]\} \cup \{H[1+], H[2+], \cdots, H[x^{'}+]\}$ $\cup\{H[+]\}$. However, not all trees in $\{H[1], H[2], \cdots, H[y^{'}]\} \cup \{H[1+], H[2+], \cdots, H[x^{'}+]\} \cup \{H[+]\}$ are child trees of H, so we need to check whether each possible child tree is actually a child tree of H.

We need some notations here. If vertex v_{i-1} has two or more children in the active tree $A(H)$, then we denote by $v_i^{'}$ the child of v_{i-1} that precedes v_i. Thus $v_i^{'}$ is the 2nd last child of v_{i-1} in $A(H)$. Similarly, for u_{i-1}, we denote by $u_i^{'}$ the 2nd last child of u_{i-1}. Note that $H(v)$ is the ordered subtree of H induced by v and all descendants of v. We now have the following lemma.

Lemma 3 *Let H be an ordered tree in $\mathcal{S}_{n,2k}$ with the right path $(v_0 = r_k, v_1, \cdots, v_x)$ and the left path $(u_0 = r_k, u_1, \cdots, u_y)$.*

(1) $H[i]$, $i \le \min\{y, k-1\}$, is a child tree of H if and only if H has at least one waiting vertex and is right empty, and for each j, $j = 1, 2, \cdots, i$, either u_{j-1} has only one child u_j in H, or $L(H(u_j^{'})) \ge L(H(u_j))$ holds in $H[i]$.

(2) $H[i+]$, $i \le \min\{x, k-1\}$, is a child tree of H if and only if H has at least one waiting vertex, and for each j, $j = 1, 2, \cdots, i$, either v_{j-1} has only one child v_j in H, or $L(H(v_j^{'})) \ge L(H(v_j))$ holds in $H[i+]$.

(3) $H[+]$ is a child tree of H if and only if H has at least two waiting vertices.

Proof : (1) Since $H \in \mathcal{S}_{n,2k}$, the left heavy condition has held in H. Then, only for vertex $u = u_0, u_1 \cdots, u_i$, $L(H(u))$ in $H[i]$ is heavier than $L(H(u))$ in H. The claim checks all of these possible changes that may destroy the left-heavy condition.

(2) (3) Omitted. ∎

If we generate each tree in $\{H[1], H[2], \cdots, H[y']\} \cup \{H[1+], H[2+], \cdots, H[x'+]\} \cup \{H[+]\}$ and check whether it is actually a child tree or not based on the lemma above, then we need considerable running time. However, we can save running time as follows. We need some definitions here.

Let H be an ordered tree in $S_{n,2k}$. We define "active at depth" in the following three cases. First, assume that H is not right empty. We say that H is *active* at depth i if (i) the right path contains a vertex v_i with depth i, (ii) v_i has two or more child vertices, and (iii) $L(H(v_{i+1}))$ is a prefix of $L(H(v'_{i+1}))$. Intuitively, if H is active at depth i, then we are copying subtree $H(v_{i+1})$ from $H(v'_{i+1})$. Then, assume that H is right empty but not left empty. We say that H is *active* at depth i if (i) the left path contains a vertex u_i with depth i, (ii) u_i has two or more child vertices, and (iii) $L(H(u_{i+1}))$ is a prefix of $L(H(u'_{i+1}))$. Then assume that H is right and left empty. We say that H is *active* at depth 0. Note that $L(H(v_1))$ is a prefix of $L(H(u_1))$.

We can show that H is always active at some depth as follows. If H is not right empty, then let j be the maximum index such that vertex v_j has two or more child vertices. Since H is not right empty, H always has such a vertex. Now since H is left-heavy and $H(v_{j+1})$ is a path, $L(H(v_{j+1}))$ is a prefix of $L(H(v'_{j+1}))$. Thus, H is active at depth j. Otherwise, H is right empty. Then if H is not left empty, in a similar manner as above, we can show that H is active at some depth. Otherwise, H is right and left empty. In this case H is active at depth 0. Therefore H is always active at some depth.

We say the *copy-depth* of H is c if H is active at depth c but not active at any depth in $\{0, 1, \cdots, c-1\}$.

Now we are going to generate all child trees of an ordered tree H in $\mathcal{S}_{n,2k}$. We have the following four cases.

We assume that H has the copy-depth c, the right path $P_{right} = (v_0 = r_k, v_1, \cdots, v_x)$ and the left path $P_{left} = (u_0 = r_k, u_1, \cdots, u_y)$.

Case 1: H has no waiting vertex. Then H corresponds to a leaf in $\mathcal{T}_{n,2k}$. Hence H has no child tree.

Case 2: Otherwise, and if H is not right empty. In this case, for $H[i]$, $i = 1, 2, \cdots, \min\{y, k-1\}$, the active leaf of $H[i]$ is on the right path of $H[i]$. So $H[i]$ is not a child tree of H. If H has two waiting vertices, then $H[+]$ is defined and is a child tree of H. The copy-depth of $H[+]$ is 0. Otherwise, H has exactly one waiting vertex and $H[+]$ is not defined. We have two subcases for $H[i+]$. Note that since Case 1 does not occur, H has a waiting vertex.

Case 2a: $L(H(v'_{c+1})) = L(H(v_{c+1}))$. (Intuitively the copy has completed.) First we show that $H[c+]$ is a child tree of H. Since H has the copy-depth c, for $j = 1, 2, \cdots, c$, $L(H(v'_j)) > L(H(v_j))$ holds in H and $L(H(v_j))$ is not a prefix of $L(H(v'_j))$. Since, for $j = 1, 2, \cdots, c$, $L(H(v_j))$ is not a prefix of $L(H(v'_j))$, $L(H(v'_j)) > L(H(v_j))$ still holds in $H[c+]$. Thus by Lemma 4.1 $H[c+]$ is a child tree of H. The copy-depth of $H[c+]$ remains at c. Similarly, $H[i+]$, $i = 1, 2, \cdots, c-1$, is a child tree of H, and the copy-depth of $H[i+]$ is i. However, for each $H[i+]$, where $i = c+1, c+2, \cdots, \min\{x, k-1\}$, the left-heavy condition is destroyed because of $L(H(v'_{c+1})) < L(H(v_{c+1}))$ in $H[i+]$. Thus, they are not child trees.

Case 2b: Otherwise. (Now $L(H(v'_{c+1})) > L(H(v_{c+1}))$ holds. Intuitively the copy has not completed yet.) Let $L(H(v'_{c+1})) = (dep(s_1), dep(s_2), \cdots, dep(s_{n'}))$, $L(H(v_{c+1})) = (dep(t_1), dep(t_2), \cdots, dep(t_{n''}))$, and set $z = dep(s_{n''+1})$. (Intuitively we are copying $H(v_{c+1})$ from $H(v'_{c+1})$ and $s_{n''+1}$ is the next vertex to be copied.) First, $H[(z-1)+]$ is a child tree of H, and the copy-depth of $H[(z-1)+]$ remains at c. Similarly, $H[1+], H[2+], \cdots, H[(z-2)+]$ are child trees of H, and we will prove in a lemma below that the copy-depth of $H[i+]$ is i for $i = 0, 1, \cdots, z-2$. For each of $H[i+]$, where $i = z, z+1, \cdots, \min\{x, k-1\}$, $L(H(v'_{c+1})) < L(H(v_{c+1}))$ holds in $H[i]$. Therefore, they are not left-heavy.

Case 3: Otherwise, and if H is not left empty. Now H is right empty and H has a waiting vertex. Let z' be the $(k+1)$-th depth in $L(H)$. Then $H[i+]$, $i = 1, 2, \cdots, z'-1$, is a child tree of H. The copy-depth of $H[i+]$ is i for $i = 1, 2, \cdots, z'-2$, and 0 for $z'-1$. On the other hand, $H[i+]$, where $i = z, z+1, \cdots, \min\{x, k-1\}$, is not a child tree of H, since $L(T(u_1)) < L(T(v_1))$ and so $H[i+]$ is not left-heavy. If H has two waiting vertices, then $H[+]$ is a child tree of H and the copy-depth of $H[+]$ is 0. Otherwise, $H[+]$ is not defined.

We have two subcases for $H[i]$. Note that H has a waiting vertex.

Case 3a: $L(H(u'_{c+1})) = L(H(u_{c+1}))$. $H[i]$, $i = 1, 2, \cdots, c$, is a child tree of H, and the copy-depth of $H[i+]$ is i. However, $H[i]$, where $i = c+1, c+2, \cdots, y$, is not a child tree of H.

Case 3b: Otherwise. Let $L(H(u'_{c+1})) = (dep(s_1), dep(s_2), \cdots, dep(s_{n'}))$, $L(H(u_{c+1})) = (dep(t_1), dep(t_2), \cdots, dep(t_{n''}))$, and set $z = dep(s_{n''+1})$. $H[1], H[2], \cdots, H[(z-1)]$ are child trees of H. The

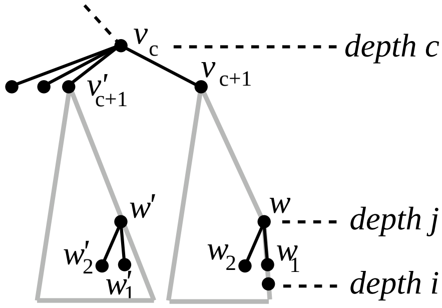

Figure 4: Illustration for Lemma 4.2.

copy-depth of $H[i]$ is i for $i = 0, 1, \cdots, z-2$, and c for $i = z-1$. For each of $H[i]$, where $i = z, z+1, \cdots, \min\{y, k-1\}$, $L(H(v'_{c+1})) < L(H(v_{c+1}))$ holds in $H[i]$, therefore they are not left-heavy.
Case 4: Otherwise. (Now H is right and left empty.) $H[i+]$, $i = 1, 2, \cdots, \min\{x, k-1\}$, is not a child tree of H. If H has two waiting vertices, then $H[+]$ is a child tree of H and the copy-depth of $H[+]$ is 0. Otherwise, $H[+]$ is not defined. $H[i]$, $i = 1, 2, \cdots, k-1$, is a child tree of H, and the copy-depth of $H[i+]$ is i.

Lemma 4 *In Case 2(b) the copy-depth of $H[i]$ is i for $i = 1, 2, \cdots, z-2$.*

Proof : For $i = 1, 2, \cdots, c$ the claim is obvious, so we assume otherwise. We can observe that the copy-depth of $H[i]$, $c+1 \le i \le z-2$, is never smaller than c, and $H[i]$ is active at i. So the copy-depth of $H[i]$ is somewhere between i and c.

Assume for contradiction that the copy-depth of $H[i]$ is $j < i$. Let $dep(w)$ be the last occurrence of depth j in $L(H[i])$. By the assumption above, w has two or more child vertices. Let w_1 be the rightmost child of w, and w_2 be the child vertex of w preceding w_1. See Fig. 4 for examples. Let w' be the vertex in $H(v'_{c+1})$ corresponding to w, and w'_1 and w'_2 be vertices in $H(v'_{c+1})$ corresponding to w_1 and w_2. (Note that we are copying $H(v_{c+1})$ from $H(v'_{c+1})$.) Now since $H \in \mathcal{S}_{n,2k}$, $L(H(w'_2)) \ge L(H(w'_1))$ holds. By the choice of i, $L(H(w'_1)) > L(H(w_1))$ holds and $L(H(w_1))$ is not a prefix of $L(H(w'_1))$. Since the copy-depth of H is c, $L(H(w'_2)) = L(H(w_2))$. Then, $L(H(w_2)) = L(H(w'_2)) \ge L(H(w'_1)) > L(H(w_1))$ holds, and $L(H(w_1))$ is not a prefix of $L(H(w'_1))$. Thus, $L(H(w_1))$ is not a prefix of $L(H(w_2))$, and the copy-depth of $H[i]$ is not j, a contradiction.

Therefore, the copy-depth of $H[i]$ is i for $i = 1, 2, \cdots, z-2$. ∎

Based on the case analysis above, we have the following algorithm.

Procedure find-all-children(T:current tree, c:copy-depth of T)
Output H { Output the difference from the preceding tree.}
if H has no waiting vertices **then return** {Case 1}
else if H is not right empty **then** {Case 2}
  if H has two waiting vertices **then find-all-children**($H[+]$, 0)
    if $L(H(v'_{c+1})) = L(H(v_{c+1}))$ **then** {Case 2a}
      for $i = 1$ **to** c **find-all-children**($H[i+]$, i)
  else {Case 2b} { $H(T(v'_{c+1})) > L(H(v_{c+1}))$ }
  { Let z be the depth of the next vertex to be copied.}
      for $i = 1$ **to** $z-2$ **find-all-children**($H[i+]$, i)
      find-all-children($H[(z-1)+]$, c)
else if H is not left empty **then** {Case 3}
  { Let z' be the $(k+1)$-th depth in $L(H)$.}
    for $i = 1$ **to** $z'-2$ **find-all-children**($H[i+]$, i)
  find-all-children($H[(z'-1)+]$, 0)
  if H has two waiting vertices **then find-all-children**($H[+]$, 0)
  if $L(H(u'_{c+1})) = L(H(u_{c+1}))$ **then** {Case 3a}
      for $i = 1$ **to** c **find-all-children**($H[i]$, i)
  else {Case 3b} { $H(T(u'_{c+1})) > L(H(u_{c+1}))$ }
  { Let z be the depth of the next vertex to be copied.}
      for $i = 1$ **to** $z-2$ **find-all-children**($H[i]$, i)

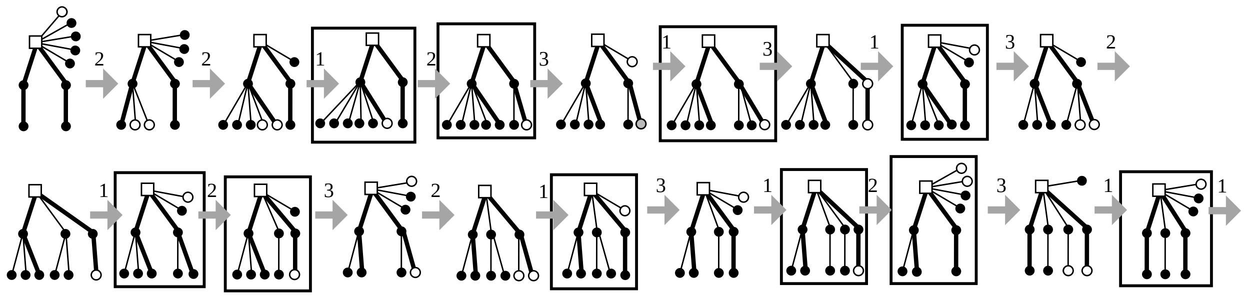

Figure 5: An execution of the algorithm for $\mathcal{T}_{10,4}$.

```
          find-all-children(H[z − 1], c)
else  {H is right empty and left empty.}
    if H has two waiting vertices then find-all-children(H[+], 0)
        for  i = 1 to k − 1 find-all-children(H[i], i)
```

Algorithm find-all-trees(n)

Output the tree H that consists of the path of length $2k$ and $(n - 2k - 1)$ of waiting vertices.
find-all-children(H, 0)

5 Modification

Our algorithm generates each tree in $O(1)$ time "average", however it often takes $O(n)$ time to return from the deep recursive call without outputting any tree after generating the tree corresponding to the last vertex in a large subtree of $\mathcal{T}_{n,d}$. Therefore, we cannot generate each tree in $O(1)$ delay. However, a simple modification improves the algorithm to generate each tree in $O(1)$ time. The algorithm is as follows.

```
Procedure find-all-children2(T, c, depth)
{ T is the current tree, c is the copy-depth of T, and depth is the depth of the recursive call.}
begin
    if  T has no waiting vertex then output T { T is a leaf.}
    else if depth is even then output T { before outputting its child trees.}
    Generate child trees T_1, T_2, ···, T_x by the method in Section 4, and
    recursively call find-all-children2 for each child tree.
    if depth is odd then Output T { after outputting its child trees.}
```

An execution of the algorithm is shown in Fig. 5. One can observe that the algorithm generates all trees so that each tree can be obtained from the preceding tree by tracing at most three edges of $\mathcal{T}_{n,k}$, each of which corresponds to an operation consisting of a deletion of a vertex and an addition of a vertex. The derived sequence of the trees is a combinatorial Gray code [J80, S97, W89] for rooted trees.

In Fig. 5 the added vertices are drawn as white circles, and the deleted, then added again, vertices are drawn as gray circles. (See the sixth tree in Fig. 5.) Each integer near an arrow mark is the number of edges in $\mathcal{T}_{n,d}$ between the two vertices corresponding to the two trees. Each tree corresponding to a vertex in $\mathcal{T}_{n,d}$ at odd depth is surrounded by a rectangle, and these trees are generated after all its child trees are generated.

Theorem 1 *Nonisomorphic trees with exactly n vertices and diameter $2k$ can be enumerated in $O(n)$ space and $O(1)$ delay.* ∎

6 The Odd Diameter Case

In this section we sketch the case where the diameter is odd.

It is known that a tree with odd diameter $2k+1$ may have many paths of length $2k+1$, but all of them share a unique edge, called *the center* of T [W01, p72].

Intuitively, by treating the edge as the root in a similar manner to the even diameter case, we can define the family tree $\mathcal{T}_{n,2k+1}$. The detail is omitted. We only show $\mathcal{T}_{10,5}$ in Fig. 6 as an example of the family tree.

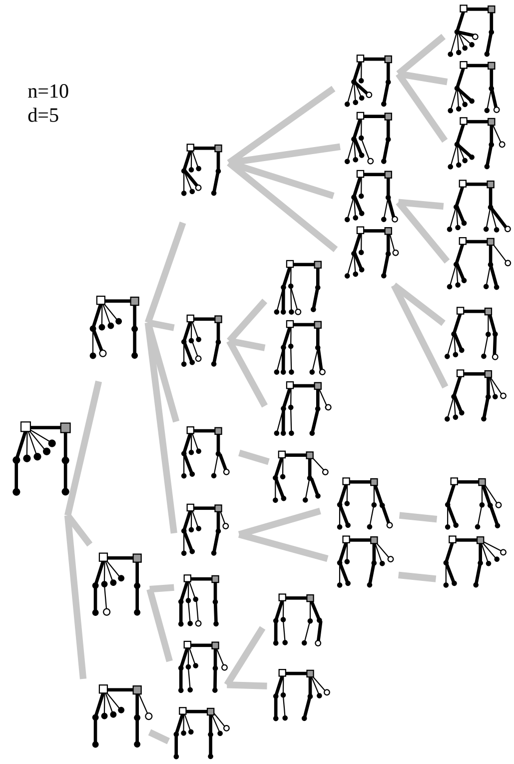

Figure 6: The family tree $\mathcal{T}_{10,5}$.

7 Conclusion

In this paper we gave a simple algorithm to generate all trees with n vertices and diameter d. The algorithm generates each tree in constant time and clarifies the family tree of the trees.

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