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完全グラフの均衡的 (C_4, C_{10}, C_{10})-Trefoil 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_{10} をそれぞれ 4 点、10 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_{10}、C_{10} からなるグラフを (C_4, C_{10}, C_{10})-trefoil という。本研究では、完全グラフ K_n を (C_4, C_{10}, C_{10})-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード : 均衡的 (C_4, C_{10}, C_{10})-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_{10}, C_{10})-Trefoil Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_{10}, C_{10})-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_{10}, C_{10})-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{10} be *the* 4*-cycle* and *the* 10*-cycle*, respectively. *The* (C_4, C_{10}, C_{10})*-trefoil* is a graph of 3 edge-disjoint cycles C_4, C_{10} and C_{10} with a common vertex and the common vertex is called *the center of the* (C_4, C_{10}, C_{10})*-trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_{10}, C_{10})-trefoils, we say that K_n *has a* (C_4, C_{10}, C_{10})*-trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_{10}, C_{10})-trefoils, we say that K_n *has a balanced* (C_4, C_{10}, C_{10})*-trefoil decomposition*

and this number is called *the replication number.*

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_n is to be known as *a balanced* (C_4, C_{10}, C_{10})*-trefoil system.*

2. Balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_n

We use the following notation for a (C_4, C_{10}, C_{10})-trefoil.

Notation. We denote a (C_4, C_{10}, C_{10})-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1$, $v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_1$, $v_1 - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_{20} - v_{21} - v_{22} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}), (v_1, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_{10}, C_{10})-trefoil decomposition if and only if $n \equiv 1$ (mod 48).

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{10}, C_{10})-trefoil decomposition. Let b be the number of (C_4, C_{10}, C_{10})-trefoils and r be the replication number. Then $b = n(n-1)/48$ and $r = 22(n-1)/48$. Among r (C_4, C_{10}, C_{10})-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{10}, C_{10})-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/48$ and $r_2 = 21(n-1)/48$. Therefore, $n \equiv 1$ (mod 48) is necessary.

(Sufficiency) Put $n = 48t + 1$. We consider 2 cases.

Case 1. $t = 1$, $n = 49$. (Example 1.) Construct a balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_{49} as follows:
$B_1 = \{(1, 35, 48, 15), (1, 2, 19, 31, 36, 13, 40, 33, 23, 4), (1, 3, 21, 32, 38, 14, 42, 34, 25, 5)\}$
$B_2 = \{(2, 36, 49, 16), (2, 3, 20, 32, 37, 14, 41, 34, 24, 5), (2, 4, 22, 33, 39, 15, 43, 35, 26, 6)\}$
$B_3 = \{(3, 37, 1, 17), (3, 4, 21, 33, 38, 15, 42, 35, 25, 6), (3, 5, 23, 34, 40, 16, 44, 36, 27, 7)\}$
$B_4 = \{(4, 38, 2, 18), (4, 5, 22, 34, 39, 16, 43, 36, 26, 7), (4, 6, 24, 35, 41, 17, 45, 37, 28, 8)\}$
$B_5 = \{(5, 39, 3, 19), (5, 6, 23, 35, 40, 17, 44, 37, 27, 8), (5, 7, 25, 36, 42, 18, 46, 38, 29, 9)\}$
$B_6 = \{(6, 40, 4, 20), (6, 7, 24, 36, 41, 18, 45, 38, 28, 9), (6, 8, 26, 37, 43, 19, 47, 39, 30, 10)\}$
$B_7 = \{(7, 41, 5, 21), (7, 8, 25, 37, 42, 19, 46, 39, 29, 10), (7, 9, 27, 38, 44, 20, 48, 40, 31, 11)\}$
$B_8 = \{(8, 42, 6, 22), (8, 9, 26, 38, 43, 20, 47, 40, 30, 11), (8, 10, 28, 39, 45, 21, 49, 41, 32, 12)\}$
$B_9 = \{(9, 43, 7, 23), (9, 10, 27, 39, 44, 21, 48, 41, 31, 12), (9, 11, 29, 40, 46, 22, 1, 42, 33, 13)\}$
$B_{10} = \{(10, 44, 8, 24), (10, 11, 28, 40, 45, 22, 49, 42, 32, 13), (10, 12, 30, 41, 47, 23, 2, 43, 34, 14)\}$
$B_{11} = \{(11, 45, 9, 25), (11, 12, 29, 41, 46, 23, 1, 43, 33, 14), (11, 13, 31, 42, 48, 24, 3, 44, 35, 15)\}$
$B_{12} = \{(12, 46, 10, 26), (12, 13, 30, 42, 47, 24, 2, 44, 34, 15), (12, 14, 32, 43, 49, 25, 4, 45, 36, 16)\}$
$B_{13} = \{(13, 47, 11, 27), (13, 14, 31, 43, 48, 25, 3, 45, 35, 16), (13, 15, 33, 44, 1, 26, 5, 46, 37, 17)\}$
$B_{14} = \{(14, 48, 12, 28), (14, 15, 32, 44, 49, 26, 4, 46, 36, 17), (14, 16, 34, 45, 2, 27, 6, 47, 38, 18)\}$
$B_{15} = \{(15, 49, 13, 29), (15, 16, 33, 45, 1, 27, 5, 47, 37, 18), (15, 17, 35, 46, 3, 28, 7, 48, 39, 19)\}$
$B_{16} = \{(16, 1, 14, 30), (16, 17, 34, 46, 2, 28, 6, 48, 38, 19), (16, 18, 36, 47, 4, 29, 8, 49, 40, 20)\}$
$B_{17} = \{(17, 2, 15, 31), (17, 18, 35, 47, 3, 29, 7, 49, 39, 20), (17, 19, 37, 48, 5, 30, 9, 1, 41, 21)\}$
$B_{18} = \{(18, 3, 16, 32), (18, 19, 36, 48, 4, 30, 8, 1, 40, 21), (18, 20, 38, 49, 6, 31, 10, 2, 42, 22)\}$

$B_{19} = \{(19,4,17,33),(19,20,37,49,5,31,9,2,41,22),(19,21,39,1,7,32,11,3,43,23)\}$
$B_{20} = \{(20,5,18,34),(20,21,38,1,6,32,10,3,42,23),(20,22,40,2,8,33,12,4,44,24)\}$
$B_{21} = \{(21,6,19,35),(21,22,39,2,7,33,11,4,43,24),(21,23,41,3,9,34,13,5,45,25)\}$
$B_{22} = \{(22,7,20,36),(22,23,40,3,8,34,12,5,44,25),(22,24,42,4,10,35,14,6,46,26)\}$
$B_{23} = \{(23,8,21,37),(23,24,41,4,9,35,13,6,45,26),(23,25,43,5,11,36,15,7,47,27)\}$
$B_{24} = \{(24,9,22,38),(24,25,42,5,10,36,14,7,46,27),(24,26,44,6,12,37,16,8,48,28)\}$
$B_{25} = \{(25,10,23,39),(25,26,43,6,11,37,15,8,47,28),(25,27,45,7,13,38,17,9,49,29)\}$
$B_{26} = \{(26,11,24,40),(26,27,44,7,12,38,16,9,48,29),(26,28,46,8,14,39,18,10,1,30)\}$
$B_{27} = \{(27,12,25,41),(27,28,45,8,13,39,17,10,49,30),(27,29,47,9,15,40,19,11,2,31)\}$
$B_{28} = \{(28,13,26,42),(28,29,46,9,14,40,18,11,1,31),(28,30,48,10,16,41,20,12,3,32)\}$
$B_{29} = \{(29,14,27,43),(29,30,47,10,15,41,19,12,2,32),(29,31,49,11,17,42,21,13,4,33)\}$
$B_{30} = \{(30,15,28,44),(30,31,48,11,16,42,20,13,3,33),(30,32,1,12,18,43,22,14,5,34)\}$
$B_{31} = \{(31,16,29,45),(31,32,49,12,17,43,21,14,4,34),(31,33,2,13,19,44,23,15,6,35)\}$
$B_{32} = \{(32,17,30,46),(32,33,1,13,18,44,22,15,5,35),(32,34,3,14,20,45,24,16,7,36)\}$
$B_{33} = \{(33,18,31,47),(33,34,2,14,19,45,23,16,6,36),(33,35,4,15,21,46,25,17,8,37)\}$
$B_{34} = \{(34,19,32,48),(34,35,3,15,20,46,24,17,7,37),(34,36,5,16,22,47,26,18,9,38)\}$
$B_{35} = \{(35,20,33,49),(35,36,4,16,21,47,25,18,8,38),(35,37,6,17,23,48,27,19,10,39)\}$
$B_{36} = \{(36,21,34,1),(36,37,5,17,22,48,26,19,9,39),(36,38,7,18,24,49,28,20,11,40)\}$
$B_{37} = \{(37,22,35,2),(37,38,6,18,23,49,27,20,10,40),(37,39,8,19,25,1,29,21,12,41)\}$
$B_{38} = \{(38,23,36,3),(38,39,7,19,24,1,28,21,11,41),(38,40,9,20,26,2,30,22,13,42)\}$
$B_{39} = \{(39,24,37,4),(39,40,8,20,25,2,29,22,12,42),(39,41,10,21,27,3,31,23,14,43)\}$
$B_{40} = \{(40,25,38,5),(40,41,9,21,26,3,30,23,13,43),(40,42,11,22,28,4,32,24,15,44)\}$
$B_{41} = \{(41,26,39,6),(41,42,10,22,27,4,31,24,14,44),(41,43,12,23,29,5,33,25,16,45)\}$
$B_{42} = \{(42,27,40,7),(42,43,11,23,28,5,32,25,15,45),(42,44,13,24,30,6,34,26,17,46)\}$
$B_{43} = \{(43,28,41,8),(43,44,12,24,29,6,33,26,16,46),(43,45,14,25,31,7,35,27,18,47)\}$
$B_{44} = \{(44,29,42,9),(44,45,13,25,30,7,34,27,17,47),(44,46,15,26,32,8,36,28,19,48)\}$
$B_{45} = \{(45,30,43,10),(45,46,14,26,31,8,35,28,18,48),(45,47,16,27,33,9,37,29,20,49)\}$
$B_{46} = \{(46,31,44,11),(46,47,15,27,32,9,36,29,19,49),(46,48,17,28,34,10,38,30,21,1)\}$
$B_{47} = \{(47,32,45,12),(47,48,16,28,33,10,37,30,20,1),(47,49,18,20,35,11,39,31,22,2)\}$
$B_{48} = \{(48,33,46,13),(48,49,17,29,34,11,38,31,21,2),(48,1,19,30,36,12,40,32,23,3)\}$
$B_{49} = \{(49,34,47,14),(49,1,18,30,35,12,39,32,22,3),(49,2,20,31,37,13,41,33,24,4)\}$.

This decomposition can be written as follows:
$B_i = \{(i,i+34,i+47,i+14),(i,i+1,i+18,i+30,i+35,i+12,i+39,i+32,i+22,i+3),$
$(i,i+2,i+20,i+31,i+37,i+13,i+41,i+33,i+24,i+4)\}$ $(i=1,2,...,49)$,
where the additions $i+x$ are taken modulo 49 with residues $1,2,...,49$.

Note. We consider the vertex set V of K_n as $V=\{1,2,...,n\}$.
The additions $i+x$ are taken modulo n with residues $1,2,...,n$.

Case 2. $t \geq 2$, $n=48t+1$. Construct tn (C_4,C_{10},C_{10})-trefoils as follows:
$B_i^{(1)} = \{(i,i+12t+1,i+45t+2,i+13t+1),(i,i+1,i+16t+2,i+28t+2,i+32t+3,i+10t+2,i+36t+3,i+30t+2,i+20t+2,i+2t+1),(i,i+2,i+16t+4,i+28t+3,i+32t+5,i+10t+3,i+36t+5,i+30t+3,i+20t+4,i+2t+2)\}$
$B_i^{(2)} = \{(i,i+12t+2,i+45t+4,i+13t+2),(i,i+3,i+16t+6,i+28t+4,i+32t+7,i+10t+4,i+36t+7,i+30t+4,i+20t+6,i+2t+3),(i,i+4,i+16t+8,i+28t+5,i+32t+9,i+10t+5,i+36t+9,i+30t+5,i+20t+8,i+2t+4)\}$
$B_i^{(3)} = \{(i,i+12t+3,i+45t+6,i+13t+3),(i,i+5,i+16t+10,i+28t+6,i+32t+11,i+10t+6,i+36t+11,i+30t+6,i+20t+10,i+2t+5),(i,i+6,i+16t+12,i+28t+7,i+32t+13,i+10t+7,i+36t+13,i+30t+7,i+20t+12,i+2t+6)\}$

...
$B_i^{(t)} = \{(i, i+13t, i+47t, i+14t), (i, i+2t-1, i+20t-2, i+30t, i+36t-1, i+12t, i+40t-1, i+32t, i+24t-2, i+4t-1), (i, i+2t, i+20t, i+30t+1, i+36t+1, i+12t+1, i+40t+1, i+32t+1, i+24t, i+4t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_n.

This completes the proof.

Example 2. A balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_{97}.
$B_i^{(1)} = \{(i, i+25, i+92, i+27), (i, i+1, i+34, i+58, i+67, i+22, i+75, i+62, i+42, i+5), (i, i+2, i+36, i+59, i+69, i+23, i+77, i+63, i+44, i+6)\}$
$B_i^{(2)} = \{(i, i+26, i+94, i+28), (i, i+3, i+38, i+60, i+71, i+24, i+79, i+64, i+46, i+7), (i, i+4, i+40, i+61, i+73, i+25, i+81, i+65, i+48, i+8)\}$
$(i = 1, 2, ..., 97)$.

Example 3. A balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_{145}.
$B_i^{(1)} = \{(i, i+37, i+137, i+40), (i, i+1, i+50, i+86, i+99, i+32, i+111, i+92, i+62, i+7), (i, i+2, i+52, i+87, i+101, i+33, i+113, i+93, i+64, i+8)\}$
$B_i^{(2)} = \{(i, i+38, i+139, i+41), (i, i+3, i+54, i+88, i+103, i+34, i+115, i+94, i+66, i+9), (i, i+4, i+56, i+89, i+105, i+35, i+117, i+95, i+68, i+10)\}$
$B_i^{(3)} = \{(i, i+39, i+141, i+42), (i, i+5, i+58, i+90, i+107, i+36, i+119, i+96, i+70, i+11), (i, i+6, i+60, i+91, i+109, i+37, i+121, i+97, i+72, i+12)\}$
$(i = 1, 2, ..., 145)$.

Example 4. A balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_{193}.
$B_i^{(1)} = \{(i, i+49, i+182, i+53), (i, i+1, i+66, i+114, i+131, i+42, i+147, i+122, i+82, i+9), (i, i+2, i+68, i+115, i+133, i+43, i+149, i+123, i+84, i+10)\}$
$B_i^{(2)} = \{(i, i+50, i+184, i+54), (i, i+3, i+70, i+116, i+135, i+44, i+151, i+124, i+86, i+11), (i, i+4, i+72, i+117, i+137, i+45, i+153, i+125, i+88, i+12)\}$
$B_i^{(3)} = \{(i, i+51, i+186, i+55), (i, i+5, i+74, i+118, i+139, i+46, i+155, i+126, i+90, i+13), (i, i+6, i+76, i+119, i+141, i+47, i+157, i+127, i+92, i+14)\}$
$B_i^{(4)} = \{(i, i+52, i+188, i+56), (i, i+7, i+78, i+120, i+143, i+48, i+159, i+128, i+94, i+15), (i, i+8, i+80, i+121, i+145, i+49, i+161, i+129, i+96, i+16)\}$
$(i = 1, 2, ..., 193)$.

Example 5. A balanced (C_4, C_{10}, C_{10})-trefoil decomposition of K_{241}.
$B_i^{(1)} = \{(i, i+61, i+227, i+66), (i, i+1, i+82, i+142, i+163, i+52, i+183, i+152, i+102, i+11), (i, i+2, i+84, i+143, i+165, i+53, i+185, i+153, i+104, i+12)\}$
$B_i^{(2)} = \{(i, i+62, i+229, i+67), (i, i+3, i+86, i+144, i+167, i+54, i+187, i+154, i+106, i+13), (i, i+4, i+88, i+145, i+169, i+55, i+189, i+155, i+108, i+14)\}$
$B_i^{(3)} = \{(i, i+63, i+231, i+68), (i, i+5, i+90, i+146, i+171, i+56, i+191, i+156, i+110, i+15), (i, i+6, i+92, i+147, i+173, i+57, i+193, i+157, i+112, i+16)\}$
$B_i^{(4)} = \{(i, i+64, i+233, i+69), (i, i+7, i+94, i+148, i+175, i+58, i+195, i+158, i+114, i+17), (i, i+8, i+96, i+149, i+177, i+59, i+197, i+159, i+116, i+18)\}$
$B_i^{(5)} = \{(i, i+65, i+235, i+70), (i, i+9, i+98, i+150, i+179, i+60, i+199, i+160, i+118, i+19), (i, i+10, i+100, i+151, i+181, i+61, i+201, i+161, i+120, i+20)\}$
$(i = 1, 2, ..., 241)$.

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