完全グラフの均衡的 (C_4, C_4, C_{10})-Trefoil 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_{10} をそれぞれ 4 点、10 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_4、C_{10} からなるグラフを (C_4, C_4, C_{10})-trefoil という。本研究では、完全グラフ K_n を (C_4, C_4, C_{10})-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード： 均衡的 (C_4, C_4, C_{10})-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_4, C_{10})-Trefoil Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_4, C_{10})-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_4, C_{10})-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{10} be *the* 4*-cycle* and *the* 10*-cycle*, respectively. *The* (C_4, C_4, C_{10})*-trefoil* is a graph of 3 edge-disjoint cycles C_4, C_4 and C_{10} with a common vertex and the common vertex is called *the center of the* (C_4, C_4, C_{10})*-trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_4, C_{10})-trefoils, we say that K_n *has a* (C_4, C_4, C_{10})*-trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_4, C_{10})-trefoils, we say that K_n *has a balanced* (C_4, C_4, C_{10})*-trefoil decomposition*

and this number is called *the replication number.*

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_4, C_{10})-trefoil decomposition of K_n is to be known as *a balanced* (C_4, C_4, C_{10})*-trefoil system.*

2. Balanced (C_4, C_4, C_{10})-trefoil decomposition of K_n

We use the following notation for a (C_4, C_4, C_{10})-trefoil.

Notation. We denote a (C_4, C_4, C_{10})-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1$, $v_1 - v_5 - v_6 - v_7 - v_1$, $v_1 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7), (v_1, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_4, C_{10})-trefoil decomposition if and only if $n \equiv 1 \pmod{36}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_4, C_{10})-trefoil decomposition. Let b be the number of (C_4, C_4, C_{10})-trefoils and r be the replication number. Then $b = n(n-1)/36$ and $r = 16(n-1)/36$. Among r (C_4, C_4, C_{10})-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_4, C_{10})-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/36$ and $r_2 = 15(n-1)/36$. Therefore, $n \equiv 1 \pmod{36}$ is necessary.

(Sufficiency) Put $n = 36t + 1$. We consider 2 cases.

Case 1. $t = 1$**,** $n = 37$**. (Example 1.)** Construct a balanced (C_4, C_4, C_{10})-trefoil decomposition of K_{37} as follows:
$B_1 = \{(1,15,22,10),(1,9,35,11),(1,2,17,23,26,8,28,24,19,3)\}$
$B_2 = \{(2,16,23,11),(2,10,36,12),(2,3,18,24,27,9,29,25,20,4)\}$
$B_3 = \{(3,17,24,12),(3,11,37,13),(3,4,19,25,28,10,30,26,21,5)\}$
$B_4 = \{(4,18,25,13),(4,12,1,14),(4,5,20,26,29,11,31,27,22,6)\}$
$B_5 = \{(5,19,26,14),(5,13,2,15),(5,6,21,27,30,12,32,28,23,7)\}$
$B_6 = \{(6,20,27,15),(6,14,3,16),(6,7,22,28,31,13,33,29,24,8)\}$
$B_7 = \{(7,21,28,16),(7,15,4,17),(7,8,23,29,32,14,34,30,25,9)\}$
$B_8 = \{(8,22,29,17),(8,16,5,18),(8,9,24,30,33,15,35,31,26,10)\}$
$B_9 = \{(9,23,30,18),(9,17,6,19),(9,10,25,31,34,16,36,32,27,11)\}$
$B_{10} = \{(10,24,31,19),(10,18,7,20),(10,11,26,32,35,17,37,33,28,12)\}$
$B_{11} = \{(11,25,32,20),(11,19,8,21),(11,12,27,33,36,18,1,34,29,13)\}$
$B_{12} = \{(12,26,33,21),(12,20,9,22),(12,13,28,34,37,19,2,35,30,14)\}$
$B_{13} = \{(13,27,34,22),(13,21,10,23),(13,14,29,35,1,20,3,36,31,15)\}$
$B_{14} = \{(14,28,35,23),(14,22,11,24),(14,15,30,36,2,21,4,37,32,16)\}$
$B_{15} = \{(15,29,36,24),(15,23,12,25),(15,16,31,37,3,22,5,1,33,17)\}$
$B_{16} = \{(16,30,37,25),(16,24,13,26),(16,17,32,1,4,23,6,2,34,18)\}$
$B_{17} = \{(17,31,1,26),(17,25,14,27),(17,18,33,2,5,24,7,3,35,19)\}$
$B_{18} = \{(18,32,2,27),(18,26,15,28),(18,19,34,3,6,25,8,4,36,20)\}$

$B_{19} = \{(19,33,3,28),(19,27,16,29),(19,20,35,4,7,26,9,5,37,21)\}$
$B_{20} = \{(20,34,4,29),(20,28,17,30),(20,21,36,5,8,27,10,6,1,22)\}$
$B_{21} = \{(21,35,5,30),(21,29,18,31),(21,22,37,6,9,28,11,7,2,23)\}$
$B_{22} = \{(22,36,6,31),(22,30,19,32),(22,23,1,7,10,29,12,8,3,24)\}$
$B_{23} = \{(23,37,7,32),(23,31,20,33),(23,24,2,8,11,30,13,9,4,25)\}$
$B_{24} = \{(24,1,8,33),(24,32,21,34),(24,25,3,9,12,31,14,10,5,26)\}$
$B_{25} = \{(25,2,9,34),(25,33,22,35),(25,26,4,10,13,32,15,11,6,27)\}$
$B_{26} = \{(26,3,10,35),(26,34,23,36),(26,27,5,11,14,33,16,12,7,28)\}$
$B_{27} = \{(27,4,11,36),(27,35,24,37),(27,28,6,12,15,34,17,13,8,29)\}$
$B_{28} = \{(28,5,12,37),(28,36,25,1),(28,29,7,13,16,35,18,14,9,30)\}$
$B_{29} = \{(29,6,13,1),(29,37,26,2),(29,30,8,14,17,36,19,15,10,31)\}$
$B_{30} = \{(30,7,14,2),(30,1,27,3),(30,31,9,15,18,37,20,16,11,32)\}$
$B_{31} = \{(31,8,15,3),(31,2,28,4),(31,32,10,16,19,1,21,17,12,33)\}$
$B_{32} = \{(32,9,16,4),(32,3,29,5),(32,33,11,17,20,2,22,18,13,34)\}$
$B_{33} = \{(33,10,17,5),(33,4,30,6),(33,34,12,18,21,3,23,19,14,35)\}$
$B_{34} = \{(34,11,18,6),(34,5,31,7),(34,35,13,19,22,4,24,20,15,36)\}$
$B_{35} = \{(35,12,19,7),(35,6,32,8),(35,36,14,20,23,5,25,21,16,37)\}$
$B_{36} = \{(36,13,20,8),(36,7,33,9),(36,37,15,21,24,6,26,22,17,1)\}$
$B_{37} = \{(37,14,21,9),(37,8,34,10),(37,1,16,22,25,7,27,23,18,2)\}$.

This decomposition can be written as follows:
$B_i = \{(i,i+14,i+21,i+9),(i,i+8,i+34,i+10),(i,i+1,i+16,i+22,i+25,i+7,i+27,i+23,i+18,i+2)\}$ $(i=1,2,...,37)$,
where the additions $i+x$ are taken modulo 37 with residues $1,2,...,37$.

Note. We consider the vertex set V of K_n as $V=\{1,2,...,n\}$.
The additions $i+x$ are taken modulo n with residues $1,2,...,n$.

Case 2. $t \geq 2$ **,** $n = 36t+1$**.** Construct tn (C_4,C_4,C_{10})-trefoils as follows:
$B_i^{(1)} = \{(i,i+6t+1,i+30t+2,i+8t+1),(i,i+6t+2,i+30t+4,i+8t+2),(i,i+1,i+14t+2,i+20t+2,i+22t+3,i+5t+2,i+24t+3,i+21t+2,i+16t+2,i+t+1)\}$
$B_i^{(2)} = \{(i,i+6t+3,i+30t+6,i+8t+3),(i,i+6t+4,i+30t+8,i+8t+4),(i,i+2,i+14t+4,i+20t+3,i+22t+5,i+5t+3,i+24t+5,i+21t+3,i+16t+4,i+t+2)\}$
$B_i^{(3)} = \{(i,i+6t+5,i+30t+10,i+8t+5),(i,i+6t+6,i+30t+12,i+8t+6),(i,i+3,i+14t+6,i+20t+4,i+22t+7,i+5t+4,i+24t+7,i+21t+4,i+16t+6,i+t+3)\}$
...
$B_i^{(t)} = \{(i,i+8t-1,i+34t-2,i+10t-1),(i,i+8t,i+34t,i+10t),(i,i+t,i+16t,i+21t+1,i+24t+1,i+6t+1,i+26t+1,i+22t+1,i+18t,i+2t)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced (C_4,C_4,C_{10})-trefoil decomposition of K_n.

This completes the proof.

Example 2. A balanced (C_4,C_4,C_{10})-trefoil decomposition of K_{73}.
$B_i^{(1)} = \{(i,i+13,i+62,i+17),(i,i+14,i+64,i+18),(i,i+1,i+30,i+42,i+47,i+12,i+51,i+44,i+34,i+3)\}$
$B_i^{(2)} = \{(i,i+15,i+66,i+19),(i,i+16,i+68,i+20),(i,i+2,i+32,i+43,i+49,i+13,i+53,i+45,i+36,i+4)\}$
$(i=1,2,...,73)$.

Example 3. A balanced (C_4, C_4, C_{10})-trefoil decomposition of K_{109}.
$B_i^{(1)} = \{(i, i+19, i+92, i+25), (i, i+20, i+94, i+26), (i, i+1, i+44, i+62, i+69, i+17, i+75, i+65, i+50, i+4)\}$
$B_i^{(2)} = \{(i, i+21, i+96, i+27), (i, i+22, i+98, i+28), (i, i+2, i+46, i+63, i+71, i+18, i+77, i+66, i+52, i+5)\}$
$B_i^{(3)} = \{(i, i+23, i+100, i+29), (i, i+24, i+102, i+30), (i, i+3, i+48, i+64, i+73, i+19, i+79, i+67, i+54, i+6)\}$
$(i = 1, 2, ..., 109)$.

Example 4. A balanced (C_4, C_4, C_{10})-trefoil decomposition of K_{145}.
$B_i^{(1)} = \{(i, i+25, i+122, i+33), (i, i+26, i+124, i+34), (i, i+1, i+58, i+82, i+91, i+22, i+99, i+86, i+66, i+5)\}$
$B_i^{(2)} = \{(i, i+27, i+126, i+35), (i, i+28, i+128, i+36), (i, i+2, i+60, i+83, i+93, i+23, i+101, i+87, i+68, i+6)\}$
$B_i^{(3)} = \{(i, i+29, i+130, i+37), (i, i+30, i+132, i+38), (i, i+3, i+62, i+84, i+95, i+24, i+103, i+88, i+70, i+7)\}$
$B_i^{(4)} = \{(i, i+31, i+134, i+39), (i, i+32, i+136, i+40), (i, i+4, i+64, i+85, i+97, i+25, i+105, i+89, i+72, i+8)\}$
$(i = 1, 2, ..., 145)$.

Example 5. A balanced (C_4, C_4, C_{10})-trefoil decomposition of K_{181}.
$B_i^{(1)} = \{(i, i+31, i+152, i+41), (i, i+32, i+154, i+42), (i, i+1, i+72, i+102, i+113, i+27, i+123, i+107, i+82, i+6)\}$
$B_i^{(2)} = \{(i, i+33, i+156, i+43), (i, i+34, i+158, i+44), (i, i+2, i+74, i+103, i+115, i+28, i+125, i+108, i+84, i+7)\}$
$B_i^{(3)} = \{(i, i+35, i+160, i+45), (i, i+36, i+162, i+46), (i, i+3, i+76, i+104, i+117, i+29, i+127, i+109, i+86, i+8)\}$
$B_i^{(4)} = \{(i, i+37, i+164, i+47), (i, i+38, i+166, i+48), (i, i+4, i+78, i+105, i+119, i+30, i+129, i+110, i+88, i+9)\}$
$B_i^{(5)} = \{(i, i+39, i+168, i+49), (i, i+40, i+170, i+50), (i, i+5, i+80, i+106, i+121, i+31, i+131, i+111, i+90, i+10)\}$
$(i = 1, 2, ..., 181)$.

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