完全対称有向グラフの均衡的 C_8-Bowtie 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_8 を８点を通る有向サイクルとする。１点を共有する辺素な２個の有向サイクル C_8、C_8 からなるグラフを C_8-bowtie という。本研究では、完全対称有向グラフ K_n^* を C_8-bowtie 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード： 均衡的 C_8-bowtie 分解; 完全対称有向グラフ; グラフ理論

Balanced C_8-Bowtie Decomposition Algorithm of Symmetric Complete Digraphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced C_8-bowtie decomposition algorithm of the symmetric complete digraph K_n^*.

Keywords: Balanced C_8-bowtie decomposition; Symmetric complete digraph; Graph theory

1. Introduction

Let K_n^* denote *the symmetric complete digraph* of n vertices. Let C_8 be *the directed* 8*-cycle. The* C_8*-bowtie* is a graph of 2 edge-disjoint directed cycles C_8, C_8 with a common vertex and the common vertex is called *the center of the* C_8*-bowtie.*
When K_n^* is decomposed into edge-disjoint sum of C_8-bowties, we say that K_n^* *has a* C_8*-bowtie decomposition.* Moreover, when every vertex of K_n^* appears in the same number of C_8-bowties, we say that K_n^* *has a balanced* C_8*-bowtie decomposition* and this number is called *the replication*

number.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_8-bowtie decomposition of K_n^* is $n \equiv 1 \pmod{16}$.

2. Balanced C_8-bowtie decomposition of K_n^*

We use the following notation for a C_8-bowtie.

Notation. We denote a C_8-bowtie passing through $v_1 \to v_2 \to v_3 \to v_4 \to v_5 \to v_6 \to v_7 \to v_8 \to v_1$, $v_1 \to v_9 \to v_{10} \to v_{11} \to v_{12} \to v_{13} \to v_{14} \to v_{15} \to v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8), (v_1, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})\}$.

We have the following theorem.

Theorem. K_n^* has a balanced C_8-bowtie decomposition if and only if $n \equiv 1 \pmod{16}$.

Proof. (Necessity) Suppose that K_n^* has a balanced C_8-bowtie decomposition. Let b be the number of C_8-bowties and r be the replication number. Then $b = n(n-1)/16$ and $r = 15(n-1)/16$. Among r C_8-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of C_8-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2r_1 + r_2 = n-1$. From these relations, $r_1 = (n-1)/16$ and $r_2 = 14(n-1)/16$. Therefore, $n \equiv 1 \pmod{16}$ is necessary.

(Sufficiency) Put $n = 16t + 1$. We consider 2 cases.

Case 1. $t = 1$, $n = 17$. (Example 1.) Construct a balanced C_8-bowtie decomposition of K_{17}^* as follows:
$B_1 = \{(1, 2, 5, 10, 17, 12, 9, 8), (1, 16, 7, 13, 15, 11, 3, 14)\}$
$B_2 = \{(2, 3, 6, 11, 1, 13, 10, 9), (2, 17, 8, 14, 16, 12, 4, 15)\}$
$B_3 = \{(3, 4, 7, 12, 2, 14, 11, 10), (3, 1, 9, 15, 17, 13, 5, 16)\}$
$B_4 = \{(4, 5, 8, 13, 3, 15, 12, 11), (4, 2, 10, 16, 1, 14, 6, 17)\}$
$B_5 = \{(5, 6, 9, 14, 4, 16, 13, 12), (5, 3, 11, 17, 2, 15, 7, 1)\}$
$B_6 = \{(6, 7, 10, 15, 5, 17, 14, 13), (6, 4, 12, 1, 3, 16, 8, 2)\}$
$B_7 = \{(7, 8, 11, 16, 6, 1, 15, 14), (7, 5, 13, 2, 4, 17, 9, 3)\}$
$B_8 = \{(8, 9, 12, 17, 7, 2, 16, 15), (8, 6, 14, 3, 5, 1, 10, 4)\}$
$B_9 = \{(9, 10, 13, 1, 8, 3, 17, 16), (9, 7, 15, 4, 6, 2, 11, 5)\}$
$B_{10} = \{(10, 11, 14, 2, 9, 4, 1, 17), (10, 8, 16, 5, 7, 3, 12, 6)\}$
$B_{11} = \{(11, 12, 15, 3, 10, 5, 2, 1), (11, 9, 17, 6, 8, 4, 13, 7)\}$
$B_{12} = \{(12, 13, 16, 4, 11, 6, 3, 2), (12, 10, 1, 7, 9, 5, 14, 8)\}$
$B_{13} = \{(13, 14, 17, 5, 12, 7, 4, 3), (13, 11, 2, 8, 10, 6, 15, 9)\}$
$B_{14} = \{(14, 15, 1, 6, 13, 8, 5, 4), (14, 12, 3, 9, 11, 7, 16, 10)\}$
$B_{15} = \{(15, 16, 2, 7, 14, 9, 6, 5), (15, 13, 4, 10, 12, 8, 17, 11)\}$
$B_{16} = \{(16, 17, 3, 8, 15, 10, 7, 6), (16, 14, 5, 11, 13, 9, 1, 12)\}$
$B_{17} = \{(17, 1, 4, 9, 16, 11, 8, 7), (17, 15, 6, 12, 14, 10, 2, 13)\}$.

This decomposition can be written as follows:
$B_i = \{(i, i+1, i+4, i+9, i+16, i+11, i+8, i+7), (i, i+15, i+6, i+12, i+14, i+10, i+2, i+13)\}$ $(i = 1, 2, ..., 17)$.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Case 2. $t \geq 2$ **,** $n = 16t + 1$**.** Construct tn C_8-bowties as follows:
$B_i^{(1)} = \{(i, i+1, i+4t+2, i+2, i+8t+3, i+8t+2, i+4t+1, i+8t+1), (i, i+2t, i+8t, i+6t-1, i+16t-1, i+14t-1, i+8t-1, i+10t)\}$
$B_i^{(2)} = \{(i, i+2, i+4t+4, i+5, i+8t+7, i+8t+5, i+4t+3, i+8t+2), (i, i+3, i+4t+6, i+8, i+8t+11, i+8t+8, i+4t+5, i+8t+3)\}$
$B_i^{(3)} = \{(i, i+4, i+4t+8, i+11, i+8t+15, i+8t+11, i+4t+7, i+8t+4), (i, i+5, i+4t+10, i+14, i+8t+19, i+8t+14, i+4t+9, i+8t+5)\}$
...
$B_i^{(t)} = \{(i, i+2t-2, i+8t-4, i+6t-7, i+16t-9, i+14t-7, i+8t-5, i+10t-2), (i, i+2t-1, i+8t-2, i+6t-4, i+16t-5, i+14t-4, i+8t-3, i+10t-1)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_8-bowtie decomposition of K_n^*.

This completes the proof.

Example 2. Balanced C_8-bowtie decomposition of K_{33}^*.
$B_i^{(1)} = \{(i, i+1, i+10, i+2, i+19, i+18, i+9, i+17), (i, i+4, i+16, i+11, i+31, i+27, i+15, i+20)\}$
$B_i^{(2)} = \{(i, i+2, i+12, i+5, i+23, i+21, i+11, i+18), (i, i+3, i+14, i+8, i+27, i+24, i+13, i+19)\}$
$(i = 1, 2, ..., 33)$.

Example 3. Balanced C_8-bowtie decomposition of K_{49}^*.
$B_i^{(1)} = \{(i, i+1, i+14, i+2, i+27, i+26, i+13, i+25), (i, i+6, i+24, i+17, i+47, i+41, i+23, i+30)\}$
$B_i^{(2)} = \{(i, i+2, i+16, i+5, i+31, i+29, i+15, i+26), (i, i+3, i+18, i+8, i+35, i+32, i+17, i+27)\}$
$B_i^{(3)} = \{(i, i+4, i+20, i+11, i+39, i+35, i+19, i+28), (i, i+5, i+22, i+14, i+43, i+38, i+21, i+29)\}$
$(i = 1, 2, ..., 49)$.

Example 4. Balanced C_8-bowtie decomposition of K_{65}^*.
$B_i^{(1)} = \{(i, i+1, i+18, i+2, i+35, i+34, i+17, i+33), (i, i+8, i+32, i+23, i+63, i+55, i+31, i+40)\}$
$B_i^{(2)} = \{(i, i+2, i+20, i+5, i+39, i+37, i+19, i+34), (i, i+3, i+22, i+8, i+43, i+40, i+21, i+35)\}$
$B_i^{(3)} = \{(i, i+4, i+24, i+11, i+47, i+43, i+23, i+36), (i, i+5, i+26, i+14, i+51, i+46, i+25, i+37)\}$
$B_i^{(4)} = \{(i, i+6, i+28, i+17, i+55, i+49, i+27, i+38), (i, i+7, i+30, i+20, i+59, i+52, i+29, i+39)\}$
$(i = 1, 2, ..., 65)$.

Example 5. Balanced C_8-bowtie decomposition of K_{81}^*.
$B_i^{(1)} = \{(i, i+1, i+22, i+2, i+43, i+42, i+21, i+41), (i, i+10, i+40, i+29, i+79, i+69, i+39, i+50)\}$
$B_i^{(2)} = \{(i, i+2, i+24, i+5, i+47, i+45, i+23, i+42), (i, i+3, i+26, i+8, i+51, i+48, i+25, i+43)\}$
$B_i^{(3)} = \{(i, i+4, i+28, i+11, i+55, i+51, i+27, i+44), (i, i+5, i+30, i+14, i+59, i+54, i+29, i+45)\}$
$B_i^{(4)} = \{(i, i+6, i+32, i+17, i+63, i+57, i+31, i+46), (i, i+7, i+34, i+20, i+67, i+60, i+33, i+47)\}$
$B_i^{(5)} = \{(i, i+8, i+36, i+23, i+71, i+63, i+35, i+48), (i, i+9, i+38, i+26, i+75, i+66, i+37, i+49)\}$
$(i = 1, 2, ..., 81)$.

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