完全多重グラフの均衡的 C_3-Eightfoil 分解アルゴリズム

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アブストラクト: グラフ理論において、グラフの分解問題は主要な研究テーマである。C_3 を３点を通るサイクルとする。１点を共有する辺素な８個の C_3 からなるグラフを C_3-eightfoil という。本研究では、完全多重グラフ λK_n を C_3-eightfoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 C_3-eightfoil 分解; 完全多重グラフ; グラフ理論

Balanced C_3-Eightfoil Decomposition Algorithm of Complete Multi-Graphs

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Abstract: In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced C_3-eightfoil decomposition algorithm of the complete multi-graph λK_n.

Keywords: Balanced C_3-eightfoil decomposition; Complete multi-graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_3 be *the 3-cycle* (or *the cycle* on 3 vertices). *The C_3-eightfoil* is a graph of 8 edge-disjoint C_3's with a common vertex and the common vertex is called *the center of the C_3-eightfoil.*
When λK_n is decomposed into edge-disjoint sum of C_3-eightfoils, we say that λK_n *has a C_3-eightfoil decomposition.* Moreover, when every vertex of λK_n appears in the same number of C_3-eightfoils, we say that λK_n *has a balanced C_3-eightfoil decomposition* and this number is called *the replication number.* This balanced C_3-eightfoil decomposition of λK_n is called *a balanced C_3-eightfoil design.*

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_3-eightfoil decomposition of λK_n is $\lambda(n-1) \equiv 0 \pmod{48}$ and $n \geq 17$. Decomposition algorithms are also given.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a C_3-bowtie system.* For combinatorial designs, see [1,4,5,15]. Another type of foil-decompositions, see [6–14].

2. Balanced C_3-eightfoil decomposition of λK_n

Notation. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. We denote a C_3-eightfoil passing through $1-2-3-1$, $1-4-5-1$, $1-6-7-1$, $1-8-9-1$, $1-10-11-1$, $1-12-13-1$, $1-14-15-1$, $1-16-17-1$ by $\{(1,2,3), (1,4,5), (1,6,7), (1,8,9), (1,10,11), (1,12,13), (1,14,15), (1,16,17)\}$. In the followings, the vertex additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Theorem 1. If λK_n has a balanced C_3-eightfoil decomposition, then $\lambda(n-1) \equiv 0 \pmod{48}$ and $n \geq 17$.
Proof. Suppose that λK_n has a balanced C_3-eightfoil decomposition. Let b be the number of C_3-eightfoils and r be the replication number. Then $b = \lambda n(n-1)/48$ and $r = 17\lambda(n-1)/48$. Among r C_3-eightfoils having a vertex v of λK_n, let r_1 and r_2 be the numbers of C_3-eightfoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $16r_1 + 2r_2 = \lambda(n-1)$. From these relations, $r_1 = \lambda(n-1)/48$ and $r_2 = 16\lambda(n-1)/48$. Thus, $\lambda(n-1) \equiv 0 \pmod{48}$. Since a C_3-eightfoil is a subgraph of λK_n, $n \geq 17$.

Note. The condition $\lambda(n-1) \equiv 0 \pmod{48}$ and $n \geq 17$ in Theorem 1 can be classified as follows:
(i) $\lambda \geq 1$ and $n \equiv 1 \pmod{48}$, $n \geq 49$
(ii) $\lambda \equiv 0 \pmod 2$ and $n \equiv 1 \pmod{24}$, $n \geq 25$
(iii) $\lambda \equiv 0 \pmod 3$ and $n \equiv 1 \pmod{16}$, $n \geq 17$
(iv) $\lambda \equiv 0 \pmod 4$ and $n \equiv 1 \pmod{12}$, $n \geq 25$
(v) $\lambda \equiv 0 \pmod 6$ and $n \equiv 1 \pmod 8$, $n \geq 17$
(vi) $\lambda \equiv 0 \pmod 8$ and $n \equiv 1 \pmod 6$, $n \geq 19$
(vii) $\lambda \equiv 0 \pmod{12}$ and $n \equiv 1 \pmod 4$, $n \geq 17$
(viii) $\lambda \equiv 0 \pmod{16}$ and $n \equiv 1 \pmod 3$, $n \geq 19$
(ix) $\lambda \equiv 0 \pmod{24}$ and $n \equiv 1 \pmod 2$, $n \geq 17$
(x) $\lambda \equiv 0 \pmod{48}$ and $n \geq 17$.

Theorem 2. If λK_n has a balanced C_3-eightfoil decomposition, then $s\lambda K_n$ has a balanced C_3-eightfoil decomposition for every s.
Proof. Obvious. Repeat s times the balanced C_3-eightfoil decomposition of λK_n.

Definition. *The C_3-t-foil* is a graph of t edge-disjoint C_3's with a common vertex and *the C_3-t-foiloid* is a multi-graph of t C_3's with a common vertex.
For example, $\{(1,2,3), (1,4,5), (1,6,7), (1,8,9)\}$ is a C_3-4-foil.
$\{(1,2,3), (1,2,4), (1,3,5), (1,4,5)\}$ is a C_3-4-foiloid.

Theorem 3. When $\lambda \geq 1$, $n \equiv 1 \pmod{48}$, and $n \geq 49$, λK_n has a balanced C_3-eightfoil decomposition.

Proof. Put $n = 48t + 1$. Construct n C_3-$8t$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+18t+1), (i, i+4t-1, i+24t), (i, i+8t, i+20t)\} \cup \{(i, i+2j, i+12t+j) \mid 1 \le j \le 4t-1\} \cup \{(i, i+1+2j, i+20t+1+j) \mid 1 \le j \le 2t-2\} \cup \{(i, i+4t-1+2j, i+22t-1+j) \mid 1 \le j \le 2t\}$.
Decompose each B_i into t C_3-eightfoils. Then they comprise a balanced C_3-eightfoil decomposition of K_n. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 4. When $\lambda \equiv 0 \pmod 2$, $n \equiv 1 \pmod{24}$, and $n \ge 25$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Put $n = 24t + 1$. Construct n C_3-$8t$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+24t-1), (i, i+3, i+24t-3), (i, i+5, i+24t-5), ..., (i, i+8t-1, i+16t+1)\}$
$\cup$ $\{(i, i+8t+1, i+8t), (i, i+8t+3, i+8t-2), (i, i+8t+5, i+8t-4), ..., (i, i+16t-1, i+2)\}$.
Decompose each B_i into t C_3-eightfoils. Then they comprise a balanced C_3-eightfoil decomposition of $2K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 5. When $\lambda \equiv 0 \pmod 3$, $n \equiv 1 \pmod{16}$, and $n \ge 17$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Put $n = 16t + 1$. Construct n C_3-$8t$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+16t), (i, i+2, i+16t-1), (i, i+3, i+16t-2), ..., (i, i+8t, i+8t+1)\}$.
Decompose each B_i into t C_3-eightfoils. Then they comprise a balanced C_3-eightfoil decomposition of $3K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 6. When $\lambda \equiv 0 \pmod 4$, $n \equiv 1 \pmod{12}$, and $n \ge 25$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider two cases.
Case 1. $n \equiv 1$ **(mod** 24**), and** $n \ge 25$. By Theorem 4 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n \equiv 13$ **(mod** 24**), and** $n \ge 37$. Put $n = 24t + 13$. First, construct n C_3-$(8t + 4)$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+24t+11), (i, i+3, i+24t+9), (i, i+5, i+24t+7), ..., (i, i+8t+3, i+16t+9)\}$
$\cup$ $\{(i, i+8t+5, i+8t+4), (i, i+8t+7, i+8t+2), (i, i+8t+9, i+8t), ..., (i, i+16t+7, i+2)\}$.
Next, put $D_i = B_i \cup B_i$. Then D_i's are C_3-$(16t + 8)$-foiloids.
Last, decompose each D_i into $2t + 1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $4K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 7. When $\lambda \equiv 0 \pmod 6$, $n \equiv 1 \pmod 8$, and $n \ge 17$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider 2 cases.
Case 1. $n \equiv 1$ **(mod** 16**), and** $n \ge 17$. By Theorem 5 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n \equiv 9$ **(mod** 16**), and** $n \ge 25$. Put $n = 16t + 9$. First, construct n C_3-$(8t + 4)$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+16t+8), (i, i+2, i+16t+7), (i, i+3, i+16t+6), ..., (i, i+8t+4, i+8t+5)\}$.
Next, put $D_i = B_i \cup B_i$. Then D_i's are C_3-$(16t + 8)$-foiloids.
Last, decompose each D_i into $2t + 1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $6K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 8. When $\lambda \equiv 0 \pmod 8$, $n \equiv 1 \pmod 6$, and $n \ge 19$, λK_n has a balanced C_3-eightfoil decomposition.

Proof. Consider 3 cases.
Case 1. $n \equiv 1 \pmod{12}$, and $n \geq 25$. By Theorem 6 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n = 19$. Construct a balanced C_3-eightfoil decomposition of $8K_{19}$ as follows: $\{(i,i+16,i+17),(i,i+12,i+15),(i,i+5,i+13),(i,i+18,i+6),(i,i+10,i+14),(i,i+11,i+2),(i,i+3,i+1),(i,i+7,i+4)\}$ $\{(i,i+1,i+17),(i,i+3,i+15),(i,i+14,i+8),(i,i+12,i+18),(i,i+9,i+4),(i,i+11,i+2),(i,i+5,i+13),(i,i+7,i+6)\}$ $\{(i,i+18,i+16),(i,i+3,i+15),(i,i+5,i+13),(i,i+7,i+6),(i,i+9,i+4),(i,i+11,i+2),(i,i+14,i+10),(i,i+8,i+17)\}$ $(i = 1,2,...,19)$. Applying Theorem 2, λK_{19} has a balanced C_3-eightfoil decomposition.
Case 3. $n \equiv 7 \pmod{12}$, and $n \geq 31$. Put $n = 12t + 7$. First, construct n C_3-$(4t+2)$-foils B_i $(i = 1,2,...,n)$ as follows:
$B_i = \{(i,i+1,i+12t+5),(i,i+2,i+12t+3),(i,i+3,i+12t+1),...,(i,i+4t+1,i+8t+5)\}$ $\cup$ $\{(i,i+4t+3,i+4t+2),(i,i+4t+5,i+4t),(i,i+4t+7,i+4t-2),...,(i,i+8t+3,i+2)\}$.
Next, put $D_i = B_i \cup B_i \cup B_i \cup B_i$. Then D_i's are C_3-$(16t+8)$-foiloids.
Last, decompose each D_i into $2t+1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 9. When $\lambda \equiv 0 \pmod{12}$, $n \equiv 1 \pmod 4$, and $n \geq 17$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider 2 cases.
Case 1. $n \equiv 1 \pmod 8$, and $n \geq 17$. By Theorem 7 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n \equiv 5 \pmod 8$, and $n \geq 21$. Put $n = 8t+5$. First, construct n C_3-$(4t+2)$-foils B_i $(i = 1,2,...,n)$ as follows:
$B_i = \{(i,i+1,i+8t+4),(i,i+2,i+8t+3),(i,i+3,i+8t+2),...,(i,i+4t+2,i+4t+3)\}$.
Next, put $D_i = B_i \cup B_i \cup B_i \cup B_i$. Then D_i's are C_3-$(16t+8)$-foiloids.
Last, decompose each D_i into $2t+1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $12K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 10. When $\lambda \equiv 0 \pmod{16}$, $n \equiv 1 \pmod 3$, and $n \geq 19$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider 3 cases.
Case 1. $n \equiv 1 \pmod 6$, and $n \geq 19$. By Theorem 8 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n = 22$. Construct a balanced C_3-eightfoil decomposition of $16K_{22}$ as follows: $\{(i,i+1,i+21),(i,i+2,i+20),(i,i+3,i+19),(i,i+4,i+18),(i,i+5,i+17),(i,i+6,i+16),(i,i+7,i+15),(i,i+13,i+14)\}$ $\{(i,i+2,i+13),(i,i+3,i+12),(i,i+4,i+11),(i,i+5,i+10),(i,i+6,i+9),(i,i+7,i+8),(i,i+1,i+21),(i,i+18,i+20)\}$ $\{(i,i+3,i+19),(i,i+4,i+18),(i,i+5,i+17),(i,i+6,i+16),(i,i+7,i+15),(i,i+1,i+14),(i,i+2,i+13),(i,i+9,i+12)\}$ $\{(i,i+4,i+11),(i,i+5,i+10),(i,i+6,i+9),(i,i+7,i+8),(i,i+1,i+21),(i,i+2,i+20),(i,i+3,i+19),(i,i+14,i+18)\}$ $\{(i,i+12,i+17),(i,i+6,i+16),(i,i+7,i+15),(i,i+1,i+14),(i,i+2,i+13),(i,i+19,i+9),(i,i+4,i+11),(i,i+5,i+10)\}$ $\{(i,i+6,i+9),(i,i+7,i+8),(i,i+1,i+21),(i,i+2,i+20),(i,i+3,i+19),(i,i+4,i+18),(i,i+5,i+17),(i,i+10,i+16)\}$ $\{(i,i+7,i+15),(i,i+1,i+9),(i,i+2,i+13),(i,i+3,i+12),(i,i+4,i+11),(i,i+5,i+10),(i,i+6,i+19),(i,i+21,i+14)\}$ $(i = 1,2,...,22)$. Applying Theorem 2, λK_{22} has a balanced C_3-eightfoil decomposition.
Case 3. $n \equiv 4 \pmod 6$, and $n \geq 28$. Put $n = 6t+4$. First, construct n C_3-$(4t+2)$-foiloids B_i $(i = 1,2,...,n)$ as follows:

$B_i = \{(i, i+1, i+6t+3), (i, i+2, i+6t+2), (i, i+3, i+6t+1), ..., (i, i+2t+1, i+4t+3)\} \cup \{(i, i+1, i+4t+2), (i, i+2, i+4t+1), (i, i+3, i+4t), ..., (i, i+2t+1, i+2t+2)\}$.
Next, put $D_i = B_i \cup B_i \cup B_i \cup B_i$. Then D_i's are C_3-$(16t+8)$-foiloids.
Last, decompose each D_i into $2t+1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Theorem 11. When $\lambda \equiv 0 \pmod{24}$, $n \equiv 1 \pmod 2$, and $n \geq 17$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider 3 cases.
Case 1. $n \equiv 1$ (mod 6), and $n \geq 19$. By Theorem 8 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n \equiv 1$ (mod 4), and $n \geq 17$. By Theorem 9 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 3. $n \equiv 3, 11$ (mod 12), and $n \geq 23$. Put $n = 12t + 8a + 3$ $(a = 0, 1)$. First, construct n C_3-$(6t+4a+1)$-foils B_i $(i = 1, 2, ..., n)$ as follows:
$B_i = \{(i, i+1, i+12t+8a+2), (i, i+2, i+12t+8a+1), (i, i+3, i+12t+8a), ..., (i, i+6t+4a+1, i+6t+4a+2)\}$.
Next, put $D_i = B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i$. Then D_i's are C_3-$(48t+32a+8)$-foiloids.
Last, decompose each D_i into $6t+4a+1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's,..., last 8 C_3's of D_i. Then they comprise a balanced C_3-eightfoil decomposition of $24K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Function. When $n \equiv 0 \pmod 2$, we use two functions $A(i)$ and $B(i)$ as follows:
When $n \equiv 0 \pmod 4$, put $n = 4t$ and let $A(i) = 2t + 2i - 1$ $(i = 1, 2, ...t)$, $A(i) = 2(i-t) - 1$ $(i = t+1, t+2, ..., 2t)$, $A(i) = 2t + 2(i - 2t)$ $(i = 2t+1, 2t+2, ..., 3t-1)$, $A(3t) = 2t$, $A(i) = 2(i - 3t)$ $(i = 3t+1, 3t+2, ..., 4t-1)$. Then $\{A(1), A(2), .., A(n-1)\} = \{1, 2, ..., n-1\}$.
When $n \equiv 2 \pmod 4$, put $n = 4t + 2$ and let $A(1) = 2t + 2$, $A(i) = 2i - 1$ $(i = 2, 3, ..., 2t+1)$, $A(i) = 2(i - 2t - 1)$ $(i = 2t+2, 2t+3, ..., 3t+1)$, $A(3t+2)$ =1, $A(i) = 2t + 2 + 2(i - 3t - 2)$ $(i = 3t+3, 3t+4, ..., 4t+1)$, and let $B(i) = 2t+2i$ $(i = 1, 2, ..., t)$, $B(t+1) = 2t+1$, $B(i) = 2(i-t-1)$ $(i = t+2, t+3, ..., 2t+1)$, $B(i) = 2t+1+2(i-2t-1)$ $(i = 2t+2, 2t+3, ..., 3t+1)$, $B(i) = 2(i-3t-1)-1$ $(i = 3t+2, 3t+3, ..., 4t+1)$. Then $\{A(1), A(2), .., A(n-1)\} = \{B(1), B(2), ..., B(n-1)\} = \{1, 2, ..., n-1\}$.

Therem 12. When $\lambda \equiv 0 \pmod{48}$ and $n \geq 17$, λK_n has a balanced C_3-eightfoil decomposition.
Proof. Consider 6 cases.
Case 1. $n \equiv 1$ (mod 2), $n \geq 17$. By Theorem 11 and Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.
Case 2. $n = 18$. Construct a balanced C_3-eightfoil decomposition of $48K_{18}$ as follows:
$\{(i, i+1, i+10), (i, i+2, i+3), (i, i+5, i+9), (i, i+17, i+16), (i, i+4, i+7), (i, i+13, i+8), (i, i+15, i+12), (i, i+6, i+11)\}$ $\{(i, i+14, i+1), (i, i+10, i+2), (i, i+3, i+5), (i, i+9, i+17), (i, i+7, i+13), (i, i+8, i+15), (i, i+12, i+6), (i, i+11, i+4)\}$ $\{(i, i+16, i+14), (i, i+1, i+10), (i, i+2, i+3), (i, i+5, i+9), (i, i+4, i+7), (i, i+13, i+8), (i, i+15, i+12), (i, i+6, i+11)\}$ $\{(i, i+17, i+16), (i, i+14, i+1), (i, i+10, i+2), (i, i+3, i+5), (i, i+7, i+13), (i, i+8, i+15), (i, i+12, i+6), (i, i+11, i+4)\}$ $\{(i, i+9, i+17), (i, i+16, i+14), (i, i+1, i+10), (i, i+2, i+3), (i, i+4, i+7), (i, i+13, i+8), (i, i+15, i+12), (i, i+6, i+11)\}$ $\{(i, i+5, i+9), (i, i+17, i+16), (i, i+14, i+1), (i, i+10, i+2), (i, i+7, i+13), (i, i+8, i+15), (i, i+12, i+6), (i, i+11, i+4)\}$ $\{(i, i+3, i+5), (i, i+9, i+17), (i, i+16, i+14), (i, i+1, i+10), (i, i+4, i+7), (i, i+13, i+8), (i, i+15, i+12), (i, i+6, i+11)\}$ $\{(i, i+2, i+3), (i, i+5, i+9), (i, i+17, i+16), (i, i+14, i+1), (i, i+7, i+13), (i, i+8, i+15), (i, i+12, i+6), (i, i+11, i+4)\}$ $\{(i, i+1, i+10), (i, i+11, i+$

$14), (i,i+17,i+7), (i,i+4,i+16), (i,i+5,i+9), (i,i+8,i+6), (i,i+2,i+12), (i,i+15,i+3)\}$ $\{(i,i+14,i+1), (i,i+10,i+11), (i,i+13,i+17), (i,i+7,i+4), (i,i+16,i+5), (i,i+9,i+8), (i,i+6,i+2), (i,i+12,i+15)\}$ $\{(i,i+3,i+14), (i,i+1,i+10), (i,i+11,i+13), (i,i+17,i+7), (i,i+4,i+16), (i,i+5,i+9), (i,i+8,i+6), (i,i+2,i+12)\}$ $\{(i,i+15,i+3), (i,i+14,i+1), (i,i+10,i+11), (i,i+13,i+17), (i,i+7,i+4), (i,i+16,i+5), (i,i+9,i+8), (i,i+6,i+2)\}$ $\{(i,i+12,i+15), (i,i+3,i+14), (i,i+10,i+2), (i,i+11,i+13), (i,i+17,i+7), (i,i+4,i+16), (i,i+5,i+9), (i,i+8,i+6)\}$ $\{(i,i+2,i+12), (i,i+15,i+3), (i,i+14,i+1), (i,i+10,i+11), (i,i+13,i+17), (i,i+7,i+4), (i,i+16,i+5), (i,i+9,i+8)\}$ $\{(i,i+6,i+2), (i,i+12,i+15), (i,i+3,i+14), (i,i+1,i+10), (i,i+11,i+13), (i,i+17,i+7), (i,i+4,i+16), (i,i+5,i+9)\}$ $\{(i,i+8,i+6), (i,i+2,i+12), (i,i+15,i+3), (i,i+9,i+1), (i,i+10,i+11), (i,i+13,i+17), (i,i+7,i+4), (i,i+16,i+5)\}$ $\{(i,i+9,i+10), (i,i+3,i+5), (i,i+8,i+17), (i,i+16,i+2), (i,i+14,i+1), (i,i+6,i+4), (i,i+12,i+15), (i,i+11,i+13)\}$ $(i=1,2,...,18)$.
Applying Theorem 2, λK_{18} has a balanced C_3-eightfoil decomposition.
Case 3. $n = 20$. Construct a balanced C_3-eightfoil decomposition of $48K_{20}$ as follows:
$\{(i,i+1,i+11), (i,i+2,i+13), (i,i+15,i+10), (i,i+4,i+17), (i,i+5,i+19), (i,i+18,i+6), (i,i+7,i+3), (i,i+12,i+14)\}$ $\{(i,i+9,i+7), (i,i+17,i+4), (i,i+11,i+12), (i,i+19,i+8), (i,i+6,i+1), (i,i+14,i+18), (i,i+3,i+15), (i,i+16,i+2)\}$ $\{(i,i+10,i+9), (i,i+18,i+6), (i,i+8,i+5), (i,i+1,i+11), (i,i+13,i+16), (i,i+3,i+15), (i,i+4,i+17), (i,i+14,i+19)\}$ $\{(i,i+6,i+1), (i,i+7,i+3), (i,i+19,i+8), (i,i+17,i+4), (i,i+10,i+9), (i,i+11,i+12), (i,i+14,i+18), (i,i+13,i+16)\}$ $\{(i,i+12,i+14), (i,i+15,i+10), (i,i+16,i+2), (i,i+9,i+7), (i,i+18,i+6), (i,i+19,i+8), (i,i+1,i+11), (i,i+4,i+17)\}$ $\{(i,i+3,i+15), (i,i+2,i+13), (i,i+11,i+12), (i,i+6,i+1), (i,i+4,i+17), (i,i+8,i+5), (i,i+14,i+18), (i,i+10,i+9)\}$ $\{(i,i+5,i+19), (i,i+12,i+14), (i,i+11,i+13), (i,i+9,i+7), (i,i+3,i+15), (i,i+16,i+2), (i,i+17,i+4), (i,i+18,i+6)\}$ $\{(i,i+8,i+3), (i,i+1,i+11), (i,i+2,i+13), (i,i+17,i+5), (i,i+18,i+6), (i,i+14,i+19), (i,i+10,i+9), (i,i+7,i+4)\}$ $\{(i,i+8,i+5), (i,i+9,i+7), (i,i+6,i+1), (i,i+11,i+12), (i,i+4,i+17), (i,i+13,i+16), (i,i+14,i+18), (i,i+15,i+10)\}$ $\{(i,i+16,i+14), (i,i+17,i+4), (i,i+18,i+6), (i,i+19,i+8), (i,i+1,i+11), (i,i+2,i+13), (i,i+3,i+15), (i,i+9,i+7)\}$ $\{(i,i+14,i+19), (i,i+6,i+1), (i,i+13,i+16), (i,i+8,i+5), (i,i+7,i+3), (i,i+10,i+9), (i,i+11,i+12), (i,i+4,i+18)\}$ $\{(i,i+7,i+3), (i,i+12,i+14), (i,i+15,i+10), (i,i+16,i+2), (i,i+17,i+4), (i,i+18,i+6), (i,i+5,i+19), (i,i+1,i+11)\}$ $\{(i,i+16,i+2), (i,i+15,i+10), (i,i+4,i+17), (i,i+5,i+19), (i,i+6,i+1), (i,i+12,i+14), (i,i+8,i+3), (i,i+9,i+7)\}$ $\{(i,i+10,i+9), (i,i+11,i+12), (i,i+7,i+3), (i,i+16,i+2), (i,i+14,i+18), (i,i+15,i+5), (i,i+19,i+8), (i,i+17,i+4)\}$ $\{(i,i+18,i+6), (i,i+3,i+16), (i,i+1,i+11), (i,i+2,i+13), (i,i+15,i+10), (i,i+9,i+7), (i,i+5,i+19), (i,i+12,i+14)\}$ $\{(i,i+7,i+3), (i,i+8,i+5), (i,i+4,i+17), (i,i+15,i+10), (i,i+11,i+12), (i,i+6,i+1), (i,i+13,i+16), (i,i+14,i+18)\}$ $\{(i,i+10,i+9), (i,i+16,i+14), (i,i+17,i+4), (i,i+18,i+6), (i,i+19,i+8), (i,i+1,i+11), (i,i+2,i+13), (i,i+3,i+15)\}$ $\{(i,i+4,i+17), (i,i+19,i+11), (i,i+13,i+16), (i,i+7,i+3), (i,i+8,i+5), (i,i+14,i+18), (i,i+10,i+9), (i,i+1,i+12)\}$ $\{(i,i+2,i+14), (i,i+3,i+16), (i,i+7,i+9), (i,i+12,i+15), (i,i+11,i+13), (i,i+17,i+4), (i,i+6,i+1), (i,i+19,i+8)\}$ $(i=1,2,...,20)$.
Applying Theorem 2, λK_{20} has a balanced C_3-eightfoil decomposition.
Case 4. $n = 22$. By Example 10.1 and Theorem 2, λK_{22} has a balanced C_3-eightfoil decomposition.
Case 5. $n \equiv 0$ **(mod 4),** $n \geq 24$. First, construct n C_3-$(n-1)$-foiloids B_i $(i = 1,2,...,n)$:
$B_i = \{(i,i+1,i+A(1)), (i,i+2,i+A(2)), (i,i+3,i+A(3)), ..., (i,i+n-1,i+A(n-1))\}$.
Next, put $D_i = B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i \cup B_i$. Then D_i's are C_3-$(8n-8)$-foiloids.
When $n = 24$, put $\sigma = (7,10)(8,11)(14,17)(31,36)(32,34)(37,42)(53,62)(54,59)(55,57)(64,65)(78,83)(79,87)(85,89)(97,105)(101,120)(102,107)(103,108)(125,130)(126,131)(138,154)(144,148)(149,155)(151,153)(167,175)(168,171)(172,178)(174,177)$.
When $n = 28$, put $\sigma = (8,12)(10,17)(13,19)(16,22)(34,43)(39,44)(40,41)(56,57)(63,66)(64,67)(70,73)$

$(71,74)(80,81)(93,98)(94,99)(117,126)(118,123)(119,121)(125,131)(144,147)(148,154)(151,153)$ $(169,177)(174,179)(175,180)(199,205)(200,202)(206,209)$.

When $n=16t$ and $n \geq 32$, put $\sigma=(8t-7,8t+1)(8t-2,8t+3)(8t-1,8t+4)(24t-3,24t+2)(24t-2,24t+3)(40t-8,40t-4)(40t-6,40t+1)(40t-3,40t+3)(40t,40t+2)(56t-8,56t-5)(56t-4,56t+2)(56t-1,56t+1)(72t-9,72t-6)(72t-8,72t-5)(72t-2,72t+1)(72t-1,72t+2)(88t-9,88t-3)(88t-8,88t-6)(88t-2,88t+1)(104t-11,104t-2)(104t-10,104t-5)(104t-9,104t-7)(104t-3,104t+1)(120t-10,120t-5)(120t-9,120t-4)$.

When $n=16t+4$ and $n \geq 36$, put $\sigma=(8t-3,8t+6)(8t-2,8t+3)(8t-1,8t+1)(8t+5,8t+9)(24t,24t+3)(24t+4,24t+10)(24t+7,24t+9)(40t+1,40t+9)(40t+6,40t+11)(40t+7,40t+12)(56t+7,56t+13)(56t+8,56t+10)(56t+14,56t+17)(72t+8,72t+12)(72t+10,72t+17)(72t+13,72t+19)(72t+16,72t+18)(88t+14,88t+19)(88t+15,88t+20)(104t+15,104t+18)(104t+16,104t+19)(104t+22,104t+25)(104t+23,104t+26)(120t+21,120t+26)(120t+22,120t+27)$.

When $n=16t+8$ and $n \geq 40$, put $\sigma=(8t-1,8t+2)(8t,8t+3)(8t+6,8t+9)(8t+7,8t+10)(24t+7,24t+13)(24t+8,24t+10)(24t+14,24t+17)(40t+13,40t+22)(40t+14,40t+19)(40t+15,40t+17)(40t+21,40t+25)(56t+22,56t+27)(56t+23,56t+28)(72t+25,72t+33)(72t+30,72t+35)(72t+31,72t+36)(88t+37,88t+42)(88t+38,88t+43)(104t+40,104t+44)(104t+42,104t+49)(104t+45,104t+51)(104t+48,104t+50)(120t+48,120t+51)(120t+52,120t+58)(120t+55,120t+57)$.

When $n=16t+12$ and $n \geq 44$, put $\sigma=(8t,8t+4)(8t+2,8t+9)(8t+5,8t+11)(8t+8,8t+10)(24t+14,24t+19)(24t+15,24t+20)(40t+23,40t+26)(40t+24,40t+27)(40t+30,40t+33)(40t+31,40t+34)(56t+37,56t+42)(56t+38,56t+43)(72t+45,72t+54)(72t+46,72t+51)(72t+47,72t+49)(72t+53,72t+57)(88t+56,88t+59)(88t+60,88t+66)(88t+63,88t+65)(104t+65,104t+73)(104t+70,104t+75)(104t+71,104t+76)(120t+79,120t+85)(120t+80,120t+82)(120t+86,120t+89)$.

Next, put $E_i=\sigma D_i$. Last, decompose each E_i into $n-1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's, ..., and last 8 C_3's of E_i. Then they comprise a balanced C_3-eightfoil decomposition of $48K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Case 6. $n \equiv 2 \pmod 4$, $n \geq 26$. First, construct n C_3-$(n-1)$-foiloids B_i and n C_3-$(n-1)$-foiloids B_i' $(i=1,2,...,n)$:

$B_i=\{(i,i+1,i+A(1)),(i,i+2,i+A(2)),(i,i+3,i+A(3)),...,(i,i+n-1,i+A(n-1))\}$

$B_i'=\{(i,i+1,i+B(1)),(i,i+2,i+B(2)),(i,i+3,i+B(3)),...,(i,i+n-1,i+B(n-1))\}$.

Next, put $D_i=B_i \cup B_i \cup B_i \cup B_i \cup B_i' \cup B_i' \cup B_i' \cup B_i'$. Then D_i's are C_3-$(8n-8)$-foiloids.

When $n=26$, put $\sigma=(3,10)(7,13)(22,27)(23,30)(29,40)(46,52)(47,49)(48,55)(54,57)(71,77)(72,74)$ $(78,81)(79,82)(96,99)(102,108)(103,107)(104,109)(110,115)(111,119)(117,121)(133,142)(134,139)$ $(135,137)(141,145)(159,165)(160,162)(164,176)(183,186)(184,187)(190,193)(191,194)$.

When $n=16t+14$ and $n \geq 30$, put $\sigma=(3,10)(7,13)(16t+8,16t+11)(16t+12,16t+17)(16t+14,16t+19)(16t+16,16t+18)(32t+22,32t+28)(32t+23,32t+25)(32t+24,32t+31)(32t+30,32t+33)(48t+37,48t+41)(48t+38,48t+44)(48t+40,48t+43)(64t+47,64t+50)(64t+48,64t+51)(72t+55,72t+61)(72t+56,72t+58)(72t+62,72t+65)(88t+65,88t+73)(88t+70,88t+75)(88t+71,88t+76)(104t+80,104t+83)(104t+84,104t+90)(104t+87,104t+89)(120t+93,120t+102)(120t+94,120t+99)(120t+95,120t+97)(120t+101,120t+105)$.

When $n=16t+2$ and $n \geq 34$, put $\sigma=(3,10)(7,13)(16t-2,16t+3)(16t-1,16t+6)(16t+5,16t+9)(32t-2,32t+4)(32t-1,32t+1)(32t,32t+7)(32t+6,32t+9)(48t-1,48t+5)(48t,48t+2)(48t+6,48t+9)(48t+7,48t+10)(64t-1,64t+2)(64t,64t+3)(72t,72t+3)(72t+4,72t+10)(72t+7,72t+9)(88t,88t+4)(88t+2,88t+9)(88t+5,88t+11)(88t+8,88t+10)(104t+5,104t+10)(104t+6,104t+11)(120t+1,120t+9)(120t+6,120t+11)(120t+7,120t+12)$.

When $n=16t+6$ and $n \geq 38$, put $\sigma=(3,10)(7,13)(16t,16t+3)(16t+4,16t+9)(16t+6,16t+11)(16t+8,16t+10)(32t+6,32t+12)(32t+7,32t+9)(32t+8,32t+15)(32t+14,32t+17)(48t+13,48t+17)(48t+14,48t+20)(48t+16,48t+19)(64t+15,64t+18)(64t+16,64t+19)(72t+21,72t+26)(72t+24,72t+27)(88t+23,88t+26)(88t+24,88t+27)(88t+30,88t+33)(88t+31,88t+34)(104t+30,104t+35)(104t+31,104t+36)(120t+32,120t+36)(120t+34,120t+41)(120t+37,120t+43)(120t+40,120t+42)$.

When $n = 16t + 10$ and $n \geq 42$, put $\sigma = (3, 10)(7, 13)(16t + 6, 16t + 11)(16t + 7, 16t + 14)(16t + 13, 16t + 17)(32t + 14, 32t + 20)(32t + 15, 32t + 17)(32t + 16, 32t + 23)(32t + 22, 32t + 25)(48t + 23, 48t + 29)(48t + 24, 48t + 26)(48t + 30, 48t + 33)(48t + 31, 48t + 34)(64t + 31, 64t + 34)(64t + 32, 64t + 35)(72t + 38, 72t + 43)(72t + 39, 72t + 44)(88t + 45, 88t + 54)(88t + 46, 88t + 51)(88t + 47, 88t+49)(88t+53, 88t+57)(104t+55, 104t+61)(104t+56, 104t+58)(104t+62, 104t+65)(120t+63, 120t + 66)(120t + 64, 120t + 67)(120t + 70, 120t + 73)(120t + 71, 120t + 74)$.
Next, put $E_i = \sigma D_i$. Last, decompose each E_i into $n - 1$ C_3-eightfoils using first 8 C_3's, next 8 C_3's, ..., and last 8 C_3's of E_i. Then they comprise a balanced C_3-eightfoil decomposition of $48K_n$. Applying Theorem 2, λK_n has a balanced C_3-eightfoil decomposition.

Therefore, we have the following main theorem and its corollary.

Main Theorem. λK_n has a balanced C_3-eightfoil decomposition if and only if $\lambda(n - 1) \equiv 0 \pmod{48}$ and $n \geq 17$.

Corollary. K_n has a balanced C_3-eightfoil decomposition if and only if $n \equiv 1 \pmod{48}$ and $n \geq 49$.

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