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非線型項書き換え系の チャーチ・ロッサ性に関連する性質について

On the Church-Rosser related properties of nonlinear term rewriting systems

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あらまし　本稿では、非線型な項書き換え系のチャーチ・ロッサ性に関連する性質（単一収束性、単一正規形性など）について論ずる。第一に、"ω-無曖昧な項書き換え系は（線形性にかかわらず）単一収束性を満たす" ことを示す。単一収束性は、従来知られているチャーチ・ロッサ性と単一正規形性の中間に位置し、停止するとはかぎらない計算の結果が宣言的意味のもとで一意であることと同値である。従来より知られている無曖昧性とω-無曖昧性の相違点は、通常のユニフィケーションのかわりに（有限グラフ表現をもつ）無限項上のユニフィケーションが用いられている点である。この結果は、ただちに、"ω-無曖昧性かつ弱停止性を満たす項書き換え系はチャーチ・ロッサ性を満たす" を導く。

第二に、（停止性を満たすとは限らない）単一収束性をもつ項書き換え系を、チャーチ・ロッサ性をもつ項書き換え系に変換する手法について、実例に基づき論じる。この変換手法は、停止しない計算の仮想的な結果を新たな関数記号により表わし、見掛け上、強停止性をみたす項書き換え系に変換する。さらに、従来より知られている完備化アルゴリズムに基づき新たな書き換え規則を加えることにより、チャーチ・ロッサ性を満たす項書き換え系に変換する。

最後に、これらの結果を、従来の関連する研究と比較する。

Abstract

This paper investigates Church-Rosser related properties of nonlinear term rewriting systems (TRS).

First, the paper investigates the relation between *Church-Rosser* related properties and nonoverlapping conditions. The hierarchy of nonoverlapping conditions corresponds to those that of unifications. *ω-nonoverlapping* is proposed in connection with *unification with infinite terms.* In contrast, the hierarchy of *Church-Rosser* related properties is based on uniqueness of computational results in various levels. *Uniquely converging*, which is an intermediate property between conventional *Church-Rosser* and *uniquely normalizing*, is proposed under the denotational semantical observation on infinite computations. The main result is, *an ω-nonoverlapping TRS is uniquely converging.*

Second, the sufficient conditions for *Church-Rosser* property of terminating TRSs are discussed. The main result is, *a weakly normalizing (i.e.any term has a reduction path which reaches to a normal form) and ω-nonoverlapping TRS is Church-Rosser.*

Third, the conversion technique from a possibly nonterminating and *uniquely converging* TRS to a *Church-Rosser* TRS is discussed by examples. The conversion technique is based on the infinite reduction detection and the conventional completion method.

Finally, these results are compared with related works.

1 *Introduction*

A Term Rewriting System (TRS), intuitively which is a set of directed equations (reduction rules), have been applied as a model for representing computational processes of equational logic and algebraic specification. The important properties of a TRS are *termination* related properties, and *Church-Rosser* (CR) related properties. Termination related properties guarantees that computations are always terminates under all/some strategies (which are said to be *strongly / weakly normalizing*). On the other hand, **CR** related properties guarantees that computational results are always unique (in various levels).

The well-known result for Church-Rosser property is that *a nonoverlapping and left linear TRS is Church-Rosser* [9], where nonoverlapping means that reductions do not interfere with each other except their variable binding environments, and left linear means each left-hand-sides of reduction rules have no repeated variables.

However, once linearity removed, **CR** is lost as well as *uniquely normalizing* (UN). This phenomena is due to following two reasons:

- Some redexes (reducible expressions) may overlap modulo reductions (though all left-hand-sides of rules are nonoverlapping). (The case of example 1)
- Some redex may not be recovered once its subterms at nonlinear variable occurrences are critically reduced. (The case of example 2)

Example 1 (*Toyama*)

$$R_1 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} d(x,x) & \to & 0 \\ d(x,f(x)) & \to & 1 \\ 2 & \to & f(2) \end{array} \right\}$$

Example2 (*Klop*)

$$R_2 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} d(x,x) & \to & 0 \\ f(x) & \to & d(x,f(x)) \\ 1 & \to & f(1) \end{array} \right\}$$

In this paper, Church-Rosser related properties of a nonlinear TRS is investigated. First, the paper shows that *an ω-nonoverlapping term rewriting systems (TRS) is uniquely converging* (UC). *ω-nonoverlapping* is different from a usual nonoverlapping condition at that *unification with infinite terms* is applied instead of usual unification with occur-check. This little stronger restriction excludes the case of example 1. On the other hand, *uniquely converging* (UC) is little weaker than **CR** such that it does not care on irregularly diverging reduction-paths (eg. $1 \xrightarrow{*}_{R_2} d(1,1) \xrightarrow{*}_{R_2} d(0,f(0)) \to_{R_2} \cdots$), but guarantees the uniqueness of results on converging reduction-paths under denotational semantical observations (eg. $1 \xrightarrow{*}_{R_2} d(1,1) \to_{R_2} 0$). This result directly implies that *a weakly normalizing (i.e.any term has a reduction path which reaches to a normal form) and ω-nonoverlapping TRS is* **CR**.

Second, the conversion technique from a **UC** (possibly nonterminating) TRS to a **CR** TRS is discussed by examples. The conversion technique is based on the infinite reduction detection and the conventional completion method.

Finally, these results are compared with related works found in literatures [4, 12, 11, 17].

2 *Reduction systems*

2.1 *Abstract reduction systems*

A *reduction system* is a structure $R = \langle A, \to \rangle$ consisting of an object set A and any binary relation $\to$ on A (i.e. $\to \subseteq A \times A$), called a *reduction relation.* A *reduction* (starting with x_0) in R is a finite or an infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The transitive closure of $\to$ is noted as $\xrightarrow{*}$.

An *equational system* associated to a reduction system $R = \langle A, \to \rangle$ is a structure $E = \langle A, =_R \rangle$ (or simply $E = \langle A, = \rangle$) consisting of A and the symmetric binary relation $\leftrightarrow_E$ (or simply $\leftrightarrow$) which is defined to be $x \leftrightarrow_E y \iff (x \to y \ \vee \ y \to x)$. An *equality* $=_E$ (or simply $=$) in E is the transitive reflexive closure of the binary relation $\leftrightarrow_R$. A sequence $x \equiv x_0 \leftrightarrow x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_n \equiv y$ is said to be an *(equational) proof* of $x = y$, and is noted as $\mathcal{P}(x = y) \equiv (x_0, x_1, x_2, \cdots, x_n)$. When apparent from context, we simply denote a proof as $\mathcal{P}$.

If an element z appears in a proof $\mathcal{P}(x = y)$ (i.e. $z \equiv x_i$ for some i in $\mathcal{P}(x = y) \equiv (x \equiv x_0, x_1, x_2, \cdots, x_n \equiv y)$), we will use the notation $z \in \mathcal{P}(x = y)$. A proof $\mathcal{P}'(x' = y')$ is said to be a sub proof of $\mathcal{P}(x = y)$ iff $x', y' \in \mathcal{P}(x = y)$ and $\mathcal{P}'(x' = y')$ is contained in $\mathcal{P}(x = y)$, and is noted as $\mathcal{P}' \subseteq \mathcal{P}$.

A *step* of a proof $\mathcal{P}(x = y)$ is denoted as $\#\mathcal{P}$. (i.e. $\#\mathcal{P} = n$ for $\mathcal{P}(x = y) \equiv (x \equiv x_0, x_1, x_2, \cdots, x_n \equiv y)$.)

Definition A set of *normal forms* $NF(R)$ of R is defined as

$NF(R) \stackrel{\text{def}}{=} \{x \in A \mid \neg \exists y \text{ s.t. } x \to y\}$

Definition $R = \langle A, \to \rangle$ is said to be *weakly normalizing* (WN), iff $\forall x \in A \ \exists y \in NF(R)$ s.t. $x \xrightarrow{*} y$. R is said to be *strongly normalizing* (SN) iff $\forall x_0 \to x_1 \to x_2 \to \cdots \to x_i \to \cdots \ \exists n$ s.t. $x_n \in NF(R)$.

Definition [9] $R = \langle A, \rightarrow \rangle$ is said to be *Church-Rosser* (**CR**) iff $\forall x, y \in A$ s.t. $x = y \Longrightarrow x \downarrow y$ (i.e. $\exists z \in A$ s.t. $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$). R is said to be *confluent* iff $\forall x, y, z \in A$ s.t. $(x \xrightarrow{*} y \ \wedge \ x \xrightarrow{*} z) \Longrightarrow y \downarrow z$.

Note that **CR** and confluence are equivalent.

Definition $R = \langle A, \rightarrow \rangle$ is said to be *uniquely normalizing with respect to equality* (**UN**) iff $\forall x, y \in NF(R)$ s.t. $x =_R y \Longrightarrow x \equiv y$. ($x \equiv y$ iff x and y are syntactically same.) $R = \langle A, \rightarrow \rangle$ is said to be *uniquely normalizing with respect to reduction* ($\mathbf{UN}^{\rightarrow}$) iff $\forall x \in A\ \forall y, z \in NF(R)$ s.t. $x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Longrightarrow y \equiv z$.

Note that $\mathbf{CR} \Longrightarrow \mathbf{UN} \Longrightarrow \mathbf{UN}^{\rightarrow}$. The converses hold when R is **WN**.

2.2 *Term rewriting systems*

Term rewriting systems are reduction systems which has a term set $T(F, V)$ as an object set A. A term set $T(F, V)$ is a set of terms where F is a set of function symbols and V is a set of variable symbols. 0-ary function symbols are also called *constants*. $T(F, V)$ may be abbreviated as simply T. The substitution θ is a map from V to $T(F, V)$ such that θ is an identity map except on a finite number of variables. The syntactical equivalence between terms M and N is denoted as $M \equiv N$.

The context C is a term in $T(F \cup \{\Box\}, V)$ where $\Box$ is a special constant named a *hole*. The notation $C[N_1, \cdots, N_n]$ is a syntax convention for the result of placing $N_1, \cdots, N_n$ in the holes of $C[, \cdots,]$ from left to right. Then, N is said to be a *subterm* of M iff $M \equiv C[N]$ for some context C having a precisely one hole. The context which has a precisely one hole is noted as $C[\ ]$. The context $C[, \cdots,]$ is said to be *trivial* iff $C \equiv \Box$.

Definition A finite set $R = \{(\alpha_i, \beta_i)\}$ of ordered pairs of two terms is said to be a *Term Rewriting System* (*TRS*) iff each α_i is not a variable and all variables in β_i appear in α_i. (Further, we assume each α_i's have no common variable names.) A *reduction* is defined on a term M as $M \rightarrow N$ iff there exists a context $C[\ ]$ and a substitution θ s.t. $M \equiv C[\theta(\alpha_i)]$ and $N \equiv C[\theta(\beta_i)]$. A subterm $M' \equiv \theta(\alpha_i)$ in M is said to be a *redex* (short for a *reducible expression*).

Definition A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be *overlapping* iff there exists a context $C[\ ]$, a nonvariable term M, and substitutions θ, θ' s.t. $\alpha_i \equiv C[M]$ and $\theta(M) \equiv \theta'(\alpha_j)$ (i.e. M and α_j are unifiable).

A TRS R is said to be *nonoverlapping* iff no pair of two rules in R are overlapping except trivial cases (i.e. $i = j \ \wedge \ C[\ ] \equiv \Box$).

Definition A reduction rule $\alpha_i \rightarrow \beta_i$ is said to be *left linear* iff any variable in α_i appears precisely once in α_i. A TRS R is said to be *left linear* iff all reduction rules in R are *left linear*. A TRS R is said to be *nonlinear* iff R is not left linear.

Remark A left linear nonoverlapping TRS is known to be confluent [9].

2.3 *Infinite trees*

For discussion on convergence, a set of infinite trees $T^{\infty}(F \cup \{\perp\}, X)$ in which a special constant $\perp$ means *undefined* is introduced.

Definition Ordering $\sqsubseteq$ on $T^{\infty}(F \cup \{\perp\}, X)$ is defined to be

$$T \sqsubseteq T' \iff T \text{ is obtained from } T' \text{ by replacing subtrees of } T' \text{ with } \perp .$$

for $\forall T, T' \in T^{\infty}(F \cup \{\perp\}, X)$.

Let $U \subseteq T^{\infty}(F \cup \{\perp\}, X)$. A pair of trees T_1, $T_2 (\in U)$ are said to be *cooperative in* U iff there exists $T \in U$ s.t. $T_1, T_2 \sqsubseteq T$. A pair of trees T_1, $T_2 (\in U)$ are said to be *individual in* U iff T_1 and T_2 are not *cooperative* in U.

$T^{\infty}(F \cup \{\perp\}, X)$ is an algebraic cpo under the ordering $\sqsubseteq$ [15]. The followings are familiar tree-related notations.

Definition An *occurrence* $occur(M, N)$ of a subterm N in a term M is defined inductively as

$$occur(M, N) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } N \equiv M \\ i \cdot u & \text{if } u = occur(N_i, N) \text{ and } M = f(N_1, \cdots, N_n) \end{cases}$$

The subterm N of M at occurrence u is denoted as M/u. (That is, $u = occur(M, N)$.) $Node(M)$ is a set of all occurrences in M (including a root occurrence ϵ). $Node^{\times}(M)$ is a set of all non-variable occurrences in M (i.e. $\{u \in Node(M) \mid M/u \text{ is not a variable}\}$).

Definition A *replacement* is noted as $T[u \leftarrow T']$ which is the replacement T/u with T' where u is an occurrence u in T. A *substitution* is noted as $T_{x \leftarrow T'} \stackrel{\text{def}}{=} T[u \leftarrow T' \mid \forall u \in occur(T, x)]$ for a variable x.

Definition The order on occurrences u, v is defined as $u \preceq v \iff \exists w$ s.t. $v = u \cdot w$. If $u \preceq v \ \wedge \ u \neq v$ then it is noted as $u \prec v$. The occurrences u, v is said to be *disjoint* and noted $u|v$ iff $u \not\leq v$ and $v \not\leq u$. Let U be any set of occurrences. A set of minimum occurrences in U is noted as $Min(U) \stackrel{\mathrm{def}}{=} \{u \in U \mid v \not< u \text{ for } \forall v \in U\}$.

3 *Uniquely converging property of nonlinear TRSs*

3.1 *Hierarchy of Church-Rosser related properties*

Church-Rosser related properties guarantee the uniqueness of results of reduction-based computations in various levels. Intuitively speaking, *Church-Rosser* guarantees the uniqueness for any reduction-paths. *Uniquely converging* guarantees for (finitely/infinitely) converging paths but doesn't care on diverging paths. *Uniquely normalizing* guarantees only for finitely converging paths (i.e. terminating paths). Each properties can be systematized by corresponding preorders on terms. For the purpose of investigation on correspondences, the retractions ω_R, ω'_R : $T^\infty(F \cup \{\bot\}, X) \rightarrow T^\infty(F \cup \{\bot\}, X)$ (which is also an embedding ω_R, ω'_R : $T(F, X) \rightarrow T^\infty(F \cup \{\bot\}, X)$) are introduced.

Definition a set of candidates of redexes $Cand_R$ is defined inductively as

- If $T \in Red_R$, then $T \in Cand_R$.
- If $T, T' \in Cand_R$, then $T[u \leftarrow T'] \in Cand_R$ for any occurrence u in T.

where Red_R is a set of all *redexes* of R.

Let $\bar{Cand_R}$ be a closure of $Cand_R$ under *Scott topology* [3] on the algebraic cpo $(T^\infty(F \cup \{\bot\}, X), \sqsubseteq)$. Then, a set of occurrences of $\bar{Cand_R}$ which appears in a term M is noted as $Candocc_R(M) \stackrel{\mathrm{def}}{=} \{u \in Node(M) \mid M/u \in \bar{Cand_R}\}$.

Definition The retractions ω_R, ω'_R : $T^\infty(F \cup \{\bot\}, X) \rightarrow T^\infty(F \cup \{\bot\}, X)$ are defined to be

$$\omega_R(M) \stackrel{\mathrm{def}}{=} M[\, u \leftarrow \bot \mid \forall u \in Min(Candocc_R(M))\,]$$

$$\omega'_R(M) \stackrel{\mathrm{def}}{=} \begin{cases} M & (M \in NF(R)) \\ \bot & (M \notin NF(R)) \end{cases}$$

Note that the retraction ω_R, ω'_R are continuous [15]. Using these retractions, hierarchical preorders are defined on terms $T(F, X)$.

Definition For $\forall x, y \in T(F, X)$,

$$x \sqsubseteq_{CR} y \stackrel{\mathrm{def}}{=} x \xrightarrow{*} y$$
$$x \sqsubseteq_{UC} y \stackrel{\mathrm{def}}{=} \omega_R(x) \sqsubseteq \omega_R(y)$$
$$x \sqsubseteq_{UN} y \stackrel{\mathrm{def}}{=} \omega'_R(x) \sqsubseteq \omega'_R(y)$$

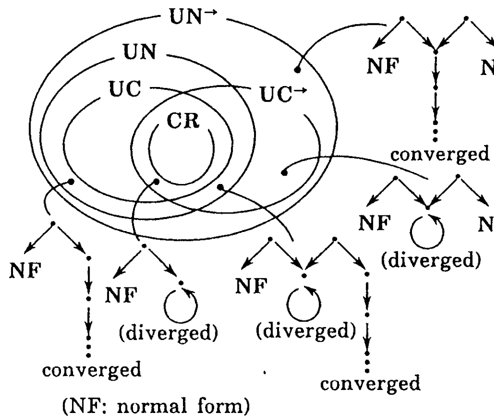

Figure 1: Relation among **CR**-related properties.

Remark $x \sqsubseteq_{CR} y \implies x \sqsubseteq_{UC} y \implies x \sqsubseteq_{UN} y$

Definition A TRS R is said to be **UC** iff $\forall y, z\ [y = z] \implies \exists x\ [(x = y = z) \wedge (y \sqsubseteq_{UC} x) \wedge (z \sqsubseteq_{UC} x)]$.

A TRS R is said to be **UC**$^{\rightarrow}$ iff $\forall x, y, z\ [(x \xrightarrow{*} y) \wedge (x \xrightarrow{*} z)] \implies \exists u\ [(x \xrightarrow{*} u) \wedge (y \sqsubseteq_{UC} u) \wedge (z \sqsubseteq_{UC} u)]$.

The hierarchy among **CR**-related properties is clarified as those that among preorders on $T(F, X)$. Their definitions based on definedness preorderings are shown in table.1.

The converses generally do not hold (see figure 1), although when R is **WN** all these properties are equivalent. Examples are also shown in figure 2 (in which a dot $\cdot$ means a term and an arrow $\rightarrow$ means a reduction). Note that **UC**$^{\rightarrow}$ is irrelevant to **UC** [16].

3.2 *Hierarchy of nonoverlapping conditions*

Let us first recall the definition of the overlapping condition.

Definition (again) A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be *overlapping* iff there exists a context $C[\,]$, a nonvariable term M, and substitutions θ, θ' s.t. $\alpha_i \equiv C[M]$ and $\theta(M) \equiv \theta'(\alpha_j)$ (i.e. M and α_j are *unifiable*).

Further, such a pair α_i and α_j is said to be an *overlapping pair.*

In this definition, usual unification with occur-check is applied. Thus, the modifications on unifications will cause variations on overlapping conditions. Actually, detailed observations show that unifications have following three

Table 1: Re-definitions of **CR**-related properties from semantical observations.

$\sqsubseteq$	$\sqsubseteq_{CR}$		$\sqsubseteq_{UC}$		$\sqsubseteq_{UN}$
with-respect-to-equality $\left(\begin{array}{l}\text{i.e. } \forall y,z\ [y=z] \\ \quad \exists x\ [(x=y=z)\ \wedge\ (y,z\sqsubseteq x)]\end{array}\right)$	CR	$\Longrightarrow$	UC	$\Longrightarrow$	UN
	$\Updownarrow$				$\Downarrow$
with-respect-to-reduction $\left(\begin{array}{l}\text{i.e. } \forall x,y,z\ [(x\xrightarrow{*}y)\ \wedge\ (x\xrightarrow{*}z)] \\ \quad \exists u\ [(x\xrightarrow{*}u)\ \wedge\ (y,z\sqsubseteq u)]\end{array}\right)$.	confluent	$\Longrightarrow$	$\mathrm{UC}^{\rightarrow}$	$\Longrightarrow$	$\mathrm{UN}^{\rightarrow}$

classes. They are,

- Unification without occur check.
- Unification with occur check.
- Unification with infinite terms (called *infinite unification*).

Unification without occur check does not care on name conflicts. Thus, even for finite terms, this is not correct for nonlinear terms. For instance, $f(x,x)$ and $f(g(y),h(y))$ are unified as $\{x = g(y),\ x = h(y)\}$. In other words, consistency of binding environments is not preserved.

In contrast, unification with occur check treats name conflicts as unification *failed*. This is correct on finite terms, but not correct on infinite terms. For instance, unification between $f(x,x)$ and $f(y,g(y))$ is failed, though it can be unified to the infinite term

$$f(g(g(g(\cdots))),g(g(g(\cdots)))).$$

The unification with infinite terms [13, 14] is between above two unifications. The difference is that expressions defining a binding environment can refer the environment itself recursively (eg. $\{x = g(x)\}$), but prohibits name conflicts on function symbols (eg. $\{x = g(x), x = h(x)\}$). Thus, for instance, $g(x,f(y,h(x)),x)$ and $g(f(h(u),v),u,u)$ are unified to a looped infinite term

$$g(f(h(f\cdots),h(f\cdots)),f(h(f\cdots),h(f\cdots)))$$

(i.e. $x = u = f(y,y),\ y = v = h(x)$).

Corresponding to these unifications, variations of overlapping conditions are similarly defined to the original definition. That is, a pair of reduction rules is said to be *ω-overlapping* (resp. *strongly overlapping*) iff unification with infinite terms (resp. unification without occur-check) is applied instead of a usual unification with occur-check in the definition of *overlapping*.

Same as the definition of *nonoverlapping*, a TRS R is said to be *ω-nonoverlapping* (resp. *strongly nonoverlapping*) iff no pair of two rules in R are ω-overlapping (resp. strongly nonoverlapping) except trivial cases (i.e. $i = j\ \wedge\ C[\ ] \equiv \Box$).

Remark If two terms are unifiable under *unification with occur-check*, unifiable under *unification with infinite terms*. If two terms are unifiable under *unification with infinite terms*, unifiable under *unification without occur-check*. Thus, *strongly nonoverlapping* implies *ω-nonoverlapping*, and *ω-nonoverlapping* implies *nonoverlapping*.

Between nonoverlapping and ω-nonoverlapping, there exists a semantical nonoverlapping condition, called *E-nonoverlapping*. Intuitively speaking, a TRS R is said to be *E-nonoverlapping* iff R is nonoverlapping *modulo* an associated equational logic E. In other words, E-nonoverlapping property is the overlapping condition under E-unification. The similar but slightly different definition of *E-nonoverlapping* is found in [10]. The difference is that *E-nonoverlapping* here does not permit a reduction at the root between an overlapping pair $\theta(M)$ and $\theta'(\alpha_j)$, whereas *E-nonoverlapping* [10] allows it.

Notation Let $\mathcal{P}(M = N)$ be a proof of $M = N$. The *boundary* $\partial\mathcal{P}$ is defined to be

$$Min\left(\left\{u \;\middle|\; \begin{array}{l}\text{A reduction at an occurrence } u\\ \text{appears in } \mathcal{P}\end{array}\right\}\right)$$

Further, we say a proof $\mathcal{P}$ is *u-preserving* iff any occurrence v appears in $\mathcal{P}$ satisfies $u \prec v$ or $u|v$ (i.e. $\forall v \in \partial\mathcal{P}$ s.t. $u \prec v \vee u|v$). We also note $M \underset{u\prec}{=} N$ iff there exists a u-preserving proof $\mathcal{P}$ of $M = N$. An occurrence u is said to be *invariant* iff $M \underset{u\prec}{=} N$.

Definition Let R be a TRS. A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be *E-overlapping* iff there exist a context $C[\ ]$, a nonvariable term M, and

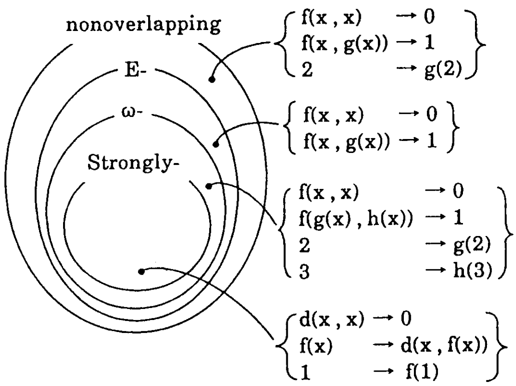

Figure 2: Relation among nonoverlapping properties.

substitutions θ, θ' s.t. $\alpha_i \equiv C[M]$ and $\theta(M) \underset{\epsilon\prec}{\equiv} \theta'(\alpha_j)$.

A TRS R is said to be *E-nonoverlapping* iff no pair of two rules in R are E-overlapping except trivial cases (i.e. $i = j \ \wedge \ C[\ ] \equiv \Box$).

Theorem 1 [16] *An ω-nonoverlapping TRS R is E-nonoverlapping.*

Figure 2 summarizes the hierarchy of nonoverlapping properties. Note that if R is left linear, all these nonoverlapping properties are equivalent although converses do not hold in general.

The next theorem characterizes E-nonoveralpping TRSs.

Theorem 2 [16] *An E-nonoverlapping TRS R is* UC *and* UC$^{\rightarrow}$.

4 *Seeking for Church-Rosser property*

In this section, the possibility and the basic ideas of completion methods for possibly nonterminating and nonlinear term rewriting systems are discussed by examples.

Basically, the completion algorithm is formalized within a framework of a proof theory. We first quote the notations and the formalizations from [2].

Let R be a strongly normalizing TRS, and $\rhd$ be a reduction ordering[1]. Since completion procedures distinguish equational rules and reduction rules, we employ a pair (E, R) as its objects, where E is a set of equational rules (represented as $x \leftrightarrow_E y$ or $x \doteq y$) and R is a set of

[1] A reduction ordering is a well-founded ordering which satisfies $M \rhd N \iff C[\theta(M)] \rhd C[\theta(N)]$ for $\forall$context $C[\ ]$, $\forall$substitution θ

C1 Orienting an equation.

$$\frac{(E\cup\{s\doteq t\}, R)}{(E, R\cup\{s\to t\})} \qquad \text{if } s \rhd t$$

C2 Adding an equational consequence.

$$\frac{(E, R)}{(E\cup\{s\doteq t\}, R)} \qquad \text{if } s \leftarrow_R u \rightarrow_R t$$

C3 Simplifying an equation.

$$\frac{(E\cup\{s\doteq t\}, R)}{(E\cup\{u\doteq t\}, R)} \qquad \text{if } s \rightarrow_R u$$

C4 Deleting a trivial equation.

$$\frac{(E\cup\{s\doteq s\}, R)}{(E, R)}.$$

Figure 3: Inference rules of $\mathcal{BC}$

reduction rules (represented as $x \rightarrow_R y$). Then, the *basic completion* $\mathcal{BC}$ consists of the *inference rules* in figure 3.

The basic completion algorithm is the (possibly infinite) sequence of deductions under a *fair* strategy[2] $(E_0, R_0) \vdash_{\mathcal{BC}} (E_1, R_1) \vdash_{\mathcal{BC}} (E_2, R_2) \vdash_{\mathcal{BC}} \cdots$ starting from $(E_0, R_0) \equiv (E_R, \phi)$ where E_R is an associated equational theory to a TRS R (For detail, refer [2]).

Beyond the limitation of **SN**, one possible direction is to treat infinite computations as *call-by-need* manner. Such a method would be realized by following steps.

- First, detect infinite reduction paths of R, and set new function symbols as the virtual limits of infinite reductions.
- Second, convert R to a pair of a strongly normalizing TRS R' and an equational theory E' which specifies the virtual limits of infinite computations. And then, apply the conventional completion algorithm starting from (E', R').

For instance, *Klop*'s example R_2 is first detected infinite computations $1 \to f(1) \to f(f(1)) \to \cdots$ and $f(x) \to d(x, f(x)) \to d(x, f(x)) \to \cdots$. First, set α and $\beta(x)$ as the virtual result values, which satisfy the equations $\alpha = f(\alpha)$ and $\beta(x) = d(x, \beta(x))$, respectively. Then, R_2 is converted to a strongly normalizing TRS R_2' (see, figure 4).

Then, the completion algorithm induces a **SN** and **CR** TRS R_2'' (see figure 5). Further, by extending ω_R as $\omega_R(\alpha) \equiv \omega_R(\beta(x)) \equiv \perp$, this conversion keeps $\omega_R(M)$ unchanged for $\forall M \in T(F, X)$.

[2] Intuitively speaking, a strategy is said to be *fair* iff any equations in E_i's and any critical pairs in R_i's will be certainly reduced by an inference rule of $\mathcal{BC}$.

$$\overset{R_2}{\left\{\begin{array}{lll} d(x,x) & \to & 0 \\ f(x) & \to & d(x,f(x)) \\ 1 & \to & f(1) \end{array}\right\}} \Longrightarrow \overset{R_2'}{\left\{\begin{array}{lll} d(x,x) & \to & 0 \\ f(x) & \to & \beta(x) \\ 1 & \to & \alpha \end{array}\right\}}$$

Figure 4: Conversion from R_2 to R_2'

$$R_2'' = \left\{\begin{array}{llllll} \beta(0) & \to & 0, & \alpha & \to & 0 \\ d(x,\beta(x)) & \to & \beta(x), & d(x,x) & \to & 0 \\ f(x) & \to & \beta(x), & 1 & \to & 0 \end{array}\right\}$$

Figure 5: The result of completion of R_2'

However, there are problematic cases such as R_3 and R_4 (see, figure 6).

For R_3, the completion algorithm successes, but produces too powerful reduction system R_3'. That is, R_3', which is **SN** and **CR**, reduces 1 to 0 whereas R_3 does not. This is caused from the fact that $h(\alpha)$ can not be distinguished from $h(h(\alpha))$ as an infinite term, where α is the virtual limit s.t. $\alpha = f(f(f(\cdots))) = h(h(h(\cdots)))$. Thus, the completion algorithm treats R_3 same as R_2.

For R_4, the completion algorithm successes, and keeps R_4 unchanged. However, R_4 is equivalent to *SKI*-combinator with δ-reduction (when f is interpreted as *apply*). Thus, R_4 is nonterminating and non-**CR**. For instance, set $C \equiv f(f(M,M),f(S,\delta))$ for $M \equiv f(f(S,f(K,f(S,I))),f(f(S,I),I)$. Then, $Y_F \xrightarrow{*} f(F,Y_F)$ where $Y_F \equiv f(f(M,M),F)$ deduces $f(f(M,M),C) \xrightarrow{*} \epsilon$, $f(C,\epsilon)$, $f(C,f(C,\epsilon)),\cdots$ (For detail, see [3] pp.399-403). The difficulty in R_4 is the automatic detection of an infinite reduction as $Y_F \xrightarrow{*} f(F,Y_F) \xrightarrow{*} f(F,f(F,Y_F)) \xrightarrow{*} \cdots$.

The completion method is attractive, but further researches are required.

$$R_3 \stackrel{\text{def}}{=} \left\{\begin{array}{lll} d(x,x) & \to & 0 \\ f(x) & \to & h(h(x)) \\ h(x) & \to & d(x,f(x)) \\ 1 & \to & f(1) \end{array}\right\}$$

$$R_4 \stackrel{\text{def}}{=} \left\{\begin{array}{lll} f(f(f(S,x),y),z) & \to & f(f(x,z),f(y,z)) \\ f(f(K,x),y) & \to & x \\ f(I,x) & \to & x \\ f(f(\delta,x),x) & \to & \epsilon \end{array}\right\}$$

Figure 6: The problematic cases, R_3, R_4

5 *Comparison with related works*

(1) Corollary 1: *An ω-nonoverlapping TRS R is* **UC** *and* **UC$^{\to}$**.

In corollary 1, the assumption *ω-nonoverlapping* is weaker than *strongly nonoverlapping*, and the result **UC** and **UC$^{\to}$** is stronger than **UN**. Thus, **corollary 1** is simple but more powerful than the following classical theorem by *Chew* [4].

Theorem (*Chew*) A TRS R is **UN** if the following conditions are met :

- R is strongly nonoverlapping.
- R is compatible.

(2) Corollary 2: *If an ω-nonoverlapping TRS R is* **WN**, *R is* **CR**.

Corollary 2 makes contrast with another classical result by *Knuth* [12].

Theorem (*Knuth*) *If a nonoverlapping TRS R is* **SN**, *R is* **CR**.

(3) *Membership conditional TRS*

The other approach to **CR**-related properties of a nonlinear TRS is found in [17]. In it, a nonoverlapping and nonlinear TRS is guaranteed to be **CR** by restricting its reductions in *call-by-value* manner when *critical*. The main theorem is,

Theorem (*Toyama*) *If a membership conditional TRS R is nonoverlapping and restricted-nonlinear, R is* **CR**.

A membership conditional TRS R is said to be *restricted-nonlinear* iff any nonlinear rule $\alpha_i \to \beta_i \in R$ reduces $\theta(\alpha_i)$ to $\theta(\beta_i)$ only when a substitution θ satisfies $\theta(x) \in NF(R)$ for all nonlinear variables x in α_i.

The example R_5 (see figure 7), which is a variation of example 1, is shown to be **CR** from the theorem (*Toyama*). In this case, the critical redex $d(2,2)$ simply diverges due to *call-by-value* manner, as $d(2,2) \to d(2,f(2)) \to d(f(2),f(2)) \to \cdots$, etc.

$$R_5 \stackrel{\text{def}}{=} \left\{\begin{array}{llll} d(x,x) & \to & 0 & \text{only if } x \in NF(R_5) \\ d(x,f(x)) & \to & 1 & \text{only if } x \in NF(R_5) \\ 2 & \to & f(2) & \end{array}\right\}$$

Figure 7: A restricted-nonlinear version of example 1

(4) *Combinator and λ-calculus with δ-reduction*

SKI-combinator with δ-reduction (eg. $\delta xx \to \epsilon$, see [3] pp.396-403) $\mathbf{CL}_{\beta\delta}$ is converted to an equivalent TRS with a function *apply* as in the previous example R_4. R_4 is ω-nonoverlapping, thus **UC**, and further **UN** (but, generally not **CR**). This easily implies the consistency of $\mathbf{CL}_{\beta\delta}$ (i.e. $\mathbf{T} \neq_{w\delta} \mathbf{F}$ where $\mathbf{T} \equiv K$, $\mathbf{F} \equiv KI$), because $K, KI \in NF(\mathrm{CL}_{\beta\delta})$ and $K \not\equiv KI$ induces $K \neq_{w\delta} KI$ from the UN property.

In contrast, the situation around λ-calulus with δ-reduction $\lambda_{\beta\delta}$ (or $\lambda_{\beta\eta\delta}$) has subtle problems on applying similar discussions. That is, reductions in λ-calculus λ_β (resp. $\lambda_{\beta\eta}$) corresponds with those that in combinator logic CL_β (resp. $CL_{\beta\eta}$) via *λ-abstraction* (see [8] pp.25). However, this correspondence is *not isomorphic* (see [8] pp.29 remark 2.25 for λ_β, and [6] pp.221 for $\lambda_{\beta\eta}$).

For instance, $M =_\beta N \implies \lambda x.M =_\beta \lambda x.N$ in λ_β calculus, but $M =_w N \not\Longrightarrow \lambda^* x.M =_w \lambda^* x.N$ in combinator logic CL_β under the λ-abstraction λ^*. Consider $M \equiv Sxyz$ and $N \equiv xz(xy)$. Clearly $M =_w N$, but $\lambda^*\, x.M \equiv S(SS(Ky))(Kz)$ and $\lambda^*\, x.N \equiv S(SI(Kz))(K(yz))$ shows that $\lambda^*\, x.M \neq_w \lambda^*\, x.N$.

Thus, the consistency of λ-calculus with δ-reduction is not induced from general discussions of the **UN** property of TRSs, but requires individual investigations. These works are briefly summarizes in [11].

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References

[1] ADJ, "Initial algebra semantics and continuous algebra, JACM, **24**,1,pp.68-95 (1977)

[2] Bachmair,L., Dershowitz,N., and Hsiang,J., "Ordering for equational proofs", *Proc.* IEEE symposium on logic in computer science, pp.346-357 (1986)

[3] Barendregt,H.P., "The Lambda Calculus, Its Syntax and Semantics", North-Holland, Amsterdam (1981)

[4] Chew,P., "Unique Normal Forms in Term Rewriting Systems with Repeated Variables", *Proc. 13th ACM STOC*, pp.7-18 (1981)

[5] Courcelle,B., "Fundamental properties of infinite trees", Theor. Comput. Sci., **25**,2,pp.95-169 (1983)

[6] Curry,H.B., Feys,R., and Craig,W., "Combinatory Logic", North-Holland, Amsterdam (1968)

[7] Dershowitz,N., and Manna,Z., "Proving termination with multiset ordering", CACM, **22**,8,pp.465-476 (1979)

[8] Hindley,J.R.., and Seldin,J.P., "Introduction to Combinators and λ-Calculus", London Mathematical Society Students Texts (1986)

[9] Huet,G., "Confluent reductions : Abstract properties and applications to term rewriting systems", JACM, **27**,4,pp.797-821 (1980)

[10] Jouannaud,J.P.,and Kirchner,H., "Completion of a set of rules modulo a set of equations", *Proc. 11th ACM POPL*, pp.83-92 (1983)

[11] Klop,J.P.,and De Vrijer,R.C., "Unique Normal Forms for Lambda Calculus with Surjective Pairing", Information and Computation, **80**, pp.97-113 (1989)

[12] Knuth,D.E.,and Bendix,P.G., "Simple word problems in universal algebra", in Leech,J.(ed.), *Computational problems in abstract algebra*, Pergamon Press, pp.263-297 (1970)

[13] Martelli,A.,and Rossi,G., "Efficient unification with infinite terms in logic programming", *Proc. FGCS 1984*, pp.202-209 (1984)

[14] Mukai,K., "A Unification Algorithm For Infinite Trees", *Proc. IJCAI 1983*, pp.547-549 (1983)

[15] Naoi,T.,and Inagaki,Y., "Semantics of Term Rewriting Systems and Free Continuous Algebra", *Trans. IEICE Japan*, **J71-D**, 6, pp.942-949 (1988) (in Japanese)

[16] Ogawa,M., and Ono,S., "On the uniquely converging property of nonlinear term rewriting systems", *in preparation* (the revised version of Technical Report COMP Vol.89, no.41, pp.61-70, *IEICE*, 1989 May)

[17] Toyama,Y., "Term rewriting systems with membership conditions", The first workshop on conditional term rewriting systems, Orsay, France (1987)