非一様並行言語に対する表示的意味モデルの構造化操作的意味論からの導出

堀田英一
NTT ソフトウェア研究所
〒180, 東京都, 武蔵野市, 緑町 3-9-11

あらまし: 並行言語の表示的意味モデルを考察する．ここで扱う言語は，エージェントの可能な動作がその現状態に依存するという意味で非一様である．最初に，操作的分岐時間モデルと操作的失敗集合モデルをラベル付き遷移システムから定義する．ここでラベル付き遷移システムは，遷移を導出するための規則の集合 $\mathcal{T}$ により定義されるものとする．次に，$\mathcal{T}$ がガード付き構造化操作的意味論形式に当てはまるとき，表示的分岐時間モデルが導出可能であり，$\mathcal{T}$ が De Simone の形式に基づくより制限された形式に当てはまるときには，表示的失敗集合モデルが導出可能であることを示す．最後に，導かれた2つの表示的モデルが対応する操作的モデルと一致することを示す．

Deriving Denotational Models for Nonuniform Concurrency from Structured Operational Semantics

(Extended Abstract)

Eiichi Horita
NTT Software Laboratories
3-9-11 Midori-Cho, Musashino-Shi, Tokyo 180 Japan

Abstract: Semantic models are studied for concurrent languages which are *nonuniform* in that possible actions of agents depend on their current state. First, two operational models each based on a *branching time* (BT) domain and a *failures domain* are defined from a *labeled transition system*, which is defined by a *set of rules* $\mathcal{T}$ for deriving transitions. Then, it is shown that a denotational model based on a BT domain (resp. a failures domain) can be derived from $\mathcal{T}$, when $\mathcal{T}$ fits into the so-called *Guarded Structured Operational Semantics* format (resp. when $\mathcal{T}$ fits into a more restricted format based on the one due to De Simone). Finally, the two denotational models are shown to be equivalent to the corresponding operational model.

1 Introduction

The problem of deriving denotational (or compositional) models for concurrent languages from a set of rules for deriving transitions has been investigated in [5], [10], [6] for BT models, and in [12] for failures models. The importance of such derivation lies in the following fact: By means it, we can obtain denotational models for a class of concurrent languages only by applying a single method, without constructing models for each individual language.[1]

All of the above works, however, have studied the problem for *uniform* languages, i.e., for languages *without* states. In the present paper, the problem is studied for concurrent languages which are *nonuniform* in that possible actions of agents depend on their current state.[2] The main contribution of the present paper is the extension of the results of [5] and [10] for BT models, and the one of [12] for failures models, to nonuniform languages.

Below an overview of the rest of this paper is given: In § 2 mathematical preliminaries are given; the main body of this paper consists of §§ 3 and 4; in § 3 (resp. in § 4) BT models (resp. failures models) are dealt with separately.

In § 3, we first define an operational BT model $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$ from a LTS $\mathcal{A}$, which is defined by a set $\mathcal{T}$ of rules for deriving transitions, along the lines of Plotkin's *Structured Operational Semantics* ([9]). (We call $\mathcal{T}$ a *transition system specification* (TSS).) Then, a denotational BT model $\mathcal{D}^{\mathrm{B}}_{\mathcal{T}}$ is derived from $\mathcal{T}$, under the assumption that $\mathcal{T}$ fits into the *Guarded Structured Operational Semantics* format (GSOS format) [2]. Finally, the equivalence between $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$ and $\mathcal{D}^{\mathrm{B}}_{\mathcal{T}}$ is established (cf. Theorem 1, the first main theorem of the present paper).

The definitions and proofs in § 3 are obtained quite straightforwardly from the counterparts (in [10], [6]) in the *uniform setting* by observing the following:

> *For* statements s, s', states σ, σ', , *and an action* a, *the transition* $(s,\sigma) \xrightarrow{a} (s',\sigma')$ *in the* nonuniform setting *corresponds to the transition* $s \xrightarrow{(\sigma,a,\sigma')} s'$ *in the uniform setting with* (σ,a,σ') *viewed as an action.* (1)

In § 4, we first define an operational failures model $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$ from the LTS $\mathcal{A}$ defined by the TSS $\mathcal{T}$. Then, a denotational failures model $\mathcal{D}^{\mathrm{F}}_{\mathcal{T}}$ is derived from $\mathcal{T}$, under the assumption that $\mathcal{T}$ fits into a more restricted format than the GSOS format; the format introduced in § 4 is based on the format due to De Simone [11] with certain additional restrictions specific to the nonuniform setting. Finally, the equivalence between $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$ and $\mathcal{D}^{\mathrm{F}}_{\mathcal{T}}$ is established.

A similar semantic equivalence problem for failures models was investigated by Vaandrager in the uniform setting [12]. However, the result of [12] in the uniform setting cannot be so directly extended to the nonuniform setting as in § 3. The approach taken in the present paper is to impose certain syntactic restrictions on the format of $\mathcal{T}$ so that semantic operations $\tilde{F}$ which are derived from $\mathcal{T}$ as interpretations of syntactic constructs should be distributive in the sense of [3] (cf. Lemma 17 in § 4.3); from the distributivity, we obtain the compositionality of $\mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}$, and thereby, the semantic equivalence (cf. Theorem 2, the second main theorem of the present paper).

2 Mathematical Preliminaries

The underlying mathematical structures of the models introduced in §§ 3 and 4, are *complete metric spaces.* The notions of *complete metric space*, *contraction*, *nonexpansive mapping*, *closure*, and *isometry* are assumed to be known. A (complete) metric space (M,d) is said to be a (complete) *ultra-metric space* iff $\forall x,y,z \in M[\ d(x,z) \le \max(d(x,y), d(y,z))\ ]$. The fact that *every contraction F from a complete metric space to itself has a unique fixed point*, known as *Banach's fixed point theorem*, is conveniently used; the unique fixed point is denoted by $\mathrm{fix}(F)$. (For the notions and fact above, the reader might consult [4].) The phrase "let $(x \in)\ X$ be $\cdots$" introduces a set X with variable x ranging over X. We use the following notation in the sequel:

Notation 1 **(1)** The standard λ-notation is used for denoting functions: For a set A, a variable x, and an expression $E(x)$, the expression $(\lambda x \in A : E(x))$ denotes the function which maps $x \in A$ to $E(x)$. We sometimes write $\langle E(x)\rangle_{x\in A}$ or $\langle E(x) : x \in A\rangle$ for $(\lambda x \in A : E(x))$.

(2) For two sets X and Y, the set of functions (resp. partial functions) from X to Y is denoted by $(X \to Y)$ or by Y^X (resp. by $(X \hookrightarrow Y)$). The set of natural numbers is denoted by ω. Each number $n \in \omega$ is identified with the set $\{i \in \omega : 0 \le i < n\}$, as usual in set theory.

(3) Let X be a topological space. For a subset $Y \subseteq X$, the *closure* of Y is denoted by Y^{cls}. The collection of closed (resp. finite) subsets of X is denoted by $\wp_{\mathrm{cl}}(X)$ (resp. by $\wp_{\mathrm{f}}(X)$). For two metric spaces X_1, X_2, let us write $X_1 \cong X_2$ to denote that *there is an isometry from* X_1 *onto* X_2. ∎

Notation 2 Let A be a set.

(1) For $n \in \omega \setminus \{0\}$ and $a_1, \cdots, a_n \in A$, the n-tuple $(a_0, \cdots, a_{n-1})$ is defined inductively in terms of *ordered pairs* as usual (cf. e.g. [4] § 1.3).

(2) The empty sequence is denoted by ϵ. The sequence consisting of $a_0, \cdots, a_{n-1} \in A$, is denoted by $\langle a_0, \cdots, a_{n-1}\rangle$. The set of finite sequences of elements of A is denoted by $A^{<\omega}$, and let $A^+ = A^{<\omega} \setminus \{\epsilon\}$. The set of finite or infinite sequences of elements of A is denoted by $A^{\le\omega}$.

[1] In the present paper, we refer to *semantic models* for programming languages simply as *models*.

[2] Here, the term '*state*' is used, to denote the state of an *agent*, which is in turn represented by a *statement* of a language.

Each sequence $w \in A^{\leq\omega}$ is regarded as a *function* whose domain is a member of $\omega \cup \{\omega\}$. Thus, the *length* of w is $\mathrm{dom}(w)$; referring to its *length* as $\mathrm{lgt}(w)$, one has $w = \langle w(i)\rangle_{i\in \mathrm{lgt}(w)} = (\lambda i \in \mathrm{lgt}(w) : w(i))$. For $a \in A$, $\nu \in \omega \cup \{\omega\}$, let $a^\nu = (\lambda i \in \nu : a)$.

(3) For $w_1 \in A^{<\omega}$, $w_2 \in A^{\leq\omega}$, let $w_1 \cdot w_2$ denote the *concatenation* of w_1 and w_2. Also, for $p_1 \subseteq A^{<\omega}$, $p_2 \subseteq A^{\leq\omega}$, let

$$p_1 \cdot p_2 = \{w_1 \cdot w_2 : w_1 \in p_1 \wedge w_2 \in p_2\}.$$

(4) For $p \subseteq A^{\leq\omega}$, $w \in A^{<\omega}$, let

$$p[w] = \{\tilde{w} \in A^{\leq\omega} : w \cdot \tilde{w} \in p\}. \blacksquare$$

A number of methods for constructing complex complete metric spaces from simple ones will be used:

Lemma 1 *Let $n \in \omega$, and (X, d), (X_i, d_i) be complete metric spaces ($i \in n$).*

(1) *The product $\prod_{i\in n}[X_i]$ is a complete metric space with a metric $\vec{d}$ defined as follows: For $\vec{x}, \vec{x}' \in \prod_{i\in n}[X_i]$, $\vec{d}(\vec{x}, \vec{x}') = \max\{d_i(\vec{x}(i), \vec{x}'(i)) : i \in n\}$.*

(2) *Let A be an arbitrary nonempty set. The product $A \times X$ is a complete metric space with a metric d' defined as follows: For $(a_1, x_1), (a_2, x_2) \in A \times X$, $d'((a_1, x_1), (a_2, x_2)) = 1$ if $a_1 \neq a_2$; otherwise, $d'((a_1, x_1), (a_2, x_2)) = (1/2) \cdot d(x_1, x_2)$.*

(3) *Let V be an arbitrary nonempty set. The function space $(V \to X)$ is a complete metric space with a metric $\tilde{d}$ defined as follows: For $f, g \in (V \to X)$, $\tilde{d}(f, g) = \sup\{d(f(v), g(v)) : v \in V\}$.* $\blacksquare$

Proof. All standard (cf. [4]). $\blacksquare$

In the sequel, products $\prod_{i\in n}[X_i]$ of complete metric spaces and function spaces $(V \to X)$ whose domains are complete metric spaces are always considered as complete metric spaces equipped with the metrics introduced above.

We use another construction method:

Definition 1 Let (X, d) be a metric space. For $x \in X$, $Z \in \wp(X)$, let $d_*(x, Z) = \inf_{z\in Z}(d(x, z))$. For $Y, Z \in \wp(X)$, the *Hausdorff distance* between Y and Z induced by d, written $d_{\mathrm{H}}(Y, Z)$, is defined by:

$$d_{\mathrm{H}}(Y, Z) = \max\{\sup_{y\in Y}(d_*(y, Z)), \sup_{z\in Z}(d_*(z, Y))\}. \blacksquare$$

The following lemma follows immediately from the definition of d_H:

Lemma 2 *Let (X, d) be metric space. Then,*

(1) $\forall Y, Z \subseteq X[\ d_{\mathrm{H}}(Y, Z) = d_{\mathrm{H}}(Y^{\mathrm{cls}}, Z^{\mathrm{cls}})\ ]$.

(2) *$(\wp_{\mathrm{cl}}(X), d_{\mathrm{H}})$ is also a metric space.* $\blacksquare$

The following fact was first established by de Bakker and Zucker [1]:

Lemma 3 *Let A be a set. Then there exists a complete ultra-metric space (P, d) such that $\mathrm{ran}(d) \subseteq [0, 1]$ and $(P, d) \cong (\wp_{\mathrm{cl}}(A \times P), d'_{\mathrm{H}})$, where d'_{H} is the Hausdorff distance induced by the metric d' (on $A \times P$) defined in Lemma 1 (2).* $\blacksquare$

In the rest of this paper, we fix a set $\mathbf{A}$ of actions, a set Σ of *states*.

3 Deriving BT Models

In this section, we first define an operational BT model $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$ from a LTS $\mathcal{A}$, which is defined by a TSS $\mathcal{T}$. Then, a denotational BT model $\mathcal{D}^{\mathrm{B}}_{\mathcal{T}}$ is derived from $\mathcal{T}$, under the assumption that $\mathcal{T}$ fits into the GSOS format. Finally, the equivalence between $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$ and $\mathcal{D}^{\mathrm{B}}_{\mathcal{T}}$ is established.

The definitions and proofs in this section are obtained quite straightforwardly from the counterparts (in [10], [6]) in the *uniform setting* by observing the fact (1) in the Introduction.

3.1 Deriving Operational BT Model

First, an operational model $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$ for concurrency based on the de Bakker-Zucker domain is induced by a LTS $\mathcal{A}$. In the rest of this subsection, we fix a LTS $\mathcal{A} = (\mathbf{S}, \Sigma, \mathbf{A}, \rightarrow)$, where $\mathbf{S}$ is a set of *states* of $\mathcal{A}$ and $\rightarrow \subseteq (\mathbf{S} \times \Sigma) \times \mathbf{A} \times (\mathbf{S} \times \Sigma)$. The LTS $\mathcal{A}$ may *infinitely branching*. We write $(s, \sigma) \xrightarrow{a} (s', \sigma')$ to denote $((s, \sigma), a, (s', \sigma')) \in \rightarrow$, as usual.[3]

Definition 2 We call a LTS $\mathcal{A} = (\mathbf{S}, \Sigma, \mathbf{A}, \rightarrow)$ *image finite* iff for each $s \in \mathbf{S}$, $(\sigma, a, \sigma') \in \Sigma \times \mathbf{A} \times \Sigma$, the set $\{s' \in \mathbf{S} : (s, \sigma) \xrightarrow{a} (s', \sigma')\}$ is finite. $\blacksquare$

Definition 3 Let ($p, q \in$) $\mathbf{P_B}$ be the complete ultra-metric space such that $\mathrm{ran}(d) \subseteq [0, 1]$ and $(\mathbf{P_B}, d) \cong (\wp_{\mathrm{cl}}((\Sigma \times \mathbf{A} \times \Sigma) \times \mathbf{P_B}), d'_{\mathrm{H}})$. (The existence of $\mathbf{P_B}$ is guaranteed by Lemma 3.) A mapping Φ_{B} from $(\mathbf{S} \to \mathbf{P_B})$ to $(\mathbf{S} \to \mathbf{P_B})$ is defined as follows: For $\mathcal{M} \in (\mathbf{S} \to \mathbf{P_B})$, $s \in \mathbf{S}$,

$$\Phi_{\mathrm{B}}(\mathcal{M})(s) = \{((\sigma, a, \sigma'), \mathcal{M}(s')) : (s, \sigma) \xrightarrow{a} (s', \sigma')\}^{\mathrm{cls}}. \blacksquare$$

Lemma 4 *The function Φ_{B} is a contraction from $(\mathbf{S} \to \mathbf{P_B})$ to $(\mathbf{S} \to \mathbf{P_B})$.* $\blacksquare$

Thus, by Banach's fixed point theorem, Φ_{B} has a unique fixed point, which is defined to be the operational BT model:

Definition 4 Let $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}} = \mathrm{fix}(\Phi_{\mathrm{B}})$. $\blacksquare$

Then the following holds by definition:

Lemma 5 *For every $s \in \mathbf{S}$,*

$$\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}[\![s]\!] = \{((\sigma, a, \sigma'), \mathcal{O}^{\mathrm{B}}_{\mathcal{A}}[\![s']\!]) : (s, \sigma) \xrightarrow{a} (s', \sigma')\}^{\mathrm{cls}}. \blacksquare$$

We define a *nonuniform* version of *strong equivalence*, called *nonuniform strong equivalence*, along the lines of Milner's definition of *strong equivalence* (cf. § 5.7 of [8]).

Definition 5 First, for $n \in \omega$, *n-nested nonuniform strong equivalence*, written $\sim^{\mathrm{NU}}_n$, is defined inductively as follows: **(i)** For every s_1, s_2, let $s_1 \sim^{\mathrm{NU}}_0 s_2$. **(ii)** For every s_1, s_2, let $s_1 \sim^{\mathrm{NU}}_{n+1} s_2$ iff $\forall (i, j) \in \{(1, 2), (2, 1)\}, \forall (\sigma, a, \sigma') \in \Sigma \times \mathbf{A} \times \Sigma[\ (s_i, \sigma) \xrightarrow{a} (s'_i, \sigma') \Rightarrow \exists s'_j[\ (s_j, \sigma) \xrightarrow{a} (s'_j, \sigma') \wedge s'_i \sim^{\mathrm{NU}}_n s'_j\ ]]$. Next, $\sim^{\mathrm{NU}}_\omega$ is defined by: $\forall s_1, s_2[\ s_1 \sim^{\mathrm{NU}}_\omega s_2 \Leftrightarrow \forall n \in \omega[\ s_1 \sim^{\mathrm{NU}}_n s_2\ ]\ ]$. $\blacksquare$

[3] We use variables s, t to ranges over $\mathbf{S}$, and σ, a for Σ and $\mathbf{A}$, respectively.

Then one has the following proposition stating that *two states are related w.r.t. nonuniform strong equivalence iff they have the same meaning in* $\mathcal{O}^{\mathrm{B}}_{\mathcal{A}}$:

Lemma 6 *For* $s_1, s_2 \in \mathbf{S}$, *one has*

$$s_1 \sim^{\mathrm{NU}}_{\omega} s_2 \Leftrightarrow \mathcal{O}^{\mathrm{B}}_{\mathcal{A}}[\![s_1]\!] = \mathcal{O}^{\mathrm{B}}_{\mathcal{A}}[\![s_2]\!]. \blacksquare$$

Proof. The same as the proof of Lemma 7 of of [6], except that the action set $\mathbf{A}$ in [6] is replaced by $\Sigma \times \mathbf{A} \times \Sigma$. ■

3.2 Deriving Denotational BT Model

In this subsection, it is shown that denotational models for concurrency based on the de Bakker-Zucker domain can be derived from the TSS $\mathcal{T}$, when $\mathcal{T}$ fits into the so-called *Guarded Structured Operational Semantics* format [2].

First, semantic operations f (on processes) associated with syntactic constructs F, are defined on the basis of a TSS $\mathcal{T}$ in the so-called *SOS format.*

The notion of *SOS format* is defined in Definition 10 below in a similar fashion to [10], with certain modification in accordance with the nonuniform setting. With the exceptions of not allowing negative antecedents and omitting the restriction of guardedness, the notion of SOS format is essentially the same as the *GSOS* format [2].

For defining *statements*, the notion of a signature is introduced:

Definition 6 A *signature* $\mathcal{S}_{\mathrm{rec}} = (\mathcal{F}_{\mathrm{rec}}, \mathrm{arity})$ is the pair of $\mathcal{F}_{\mathrm{rec}}$ (a set of function symbols) and a mapping $\mathrm{arity}(\cdot) : \mathcal{F}_{\mathrm{rec}} \to \omega$ which maps each function symbol to its *arity*.[4] It is assumed that a set of symbols $(Z \in)$ $\mathcal{Z}$ is predefined, and $\mathcal{F}_{\mathrm{rec}}$ includes $\mathcal{Z}$; the members of $\mathcal{Z}$ are reserved for the names of *recursive statements.* Let us put $\mathcal{F} = \mathcal{F}_{\mathrm{rec}} \setminus \mathcal{Z}$, and $\mathcal{S} = (\mathcal{F}, \mathrm{arity})$. For each $r \in \omega$, let $\mathcal{F}^{(r)} = \{F \in \mathcal{F} : \mathrm{arity}(F) = r\}$. ■

The set of statements is defined as follows:

Definition 7 **(1)** Let $\mathcal{X}$ be a set of variables ranging over statements. Let $\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})$ be defined to be the set of *terms* generated by $\mathcal{S}_{\mathrm{rec}}$, $\mathcal{X}$. Formally, the set $(S \in)$ $\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})$ is defined by the following BNF grammar:

$$S ::= X \mid F(S_0, \cdots, S_{r-1}) \mid Z,$$

where $r \in \omega$, $F \in \mathcal{F}^{(r)}$, and $Z \in \mathcal{Z}$.

(2) For $S \in \Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})$, the set of *variables* contained in S, denoted by $\mathrm{Var}(S)$, is defined as usual. For $\mathcal{Y} \subseteq \mathcal{X}$, let $\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})^{\mathcal{Y}} = \{S \in \Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}) : \mathrm{Var}(U) \subseteq \mathcal{Y}\}$. Elements of $\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})^{\emptyset}$, viz. terms containing no variable are said to be *closed. Statements* are defined to closed terms.

(3) Let $\Lambda(\mathcal{S}, \mathcal{X})$ be the set of terms generated by $\mathcal{S}$, $\mathcal{X}$. Further let $\Lambda^1(\mathcal{S}, \mathcal{X})$ be the set of elements S of $\Lambda(\mathcal{S}, \mathcal{X})$ such that for every $X \in \mathcal{X}$ there is at most one occurrence of X in S. ■

In the sequel, *syntactic identity* is denoted by '$\equiv$'. We introduce the notions of *syntactic* and *semantic evaluations* below.

[4] We refer to function symbols with arity 0 as *constants.*

Definition 8 A *syntactic valuation* is a partial function from $\mathcal{X}$ to $\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})$. Thus, $(\zeta \in)$ $(\mathcal{X} \hookrightarrow \Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}))$ is the set of syntactic valuations. For $S \in \Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})$, $\zeta \in (\mathcal{X} \hookrightarrow \Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}))$, the expression $S[\zeta]$ denotes the result of simultaneously replacing X in S with $\zeta(X)$ $(X \in \mathrm{dom}(\zeta))$. For a set I, $\vec{X} \in \mathcal{X}^I$, $\vec{S} \in (\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}))^I$, let $\vec{S}/\vec{X}$ be the syntactic valuation $\{(\vec{X}(i), \vec{S}(i)) : i \in I\}$, which maps $\vec{X}(i)$ to $\vec{S}(i)$ $(i \in I)$. Thus $S[\vec{S}/\vec{X}]$ denotes the result of replacing $\vec{X}(i)$ by $\vec{S}(i)$ $(i \in n)$. ■

Definition 9 Let J be an *interpretation* of the signature $\mathcal{S}$ with some domain $\mathbf{D}$. (Here an *interpretation* refers to a mapping which maps each syntactic construct to a semantic operation with an appropriate domain and range in accordance with tis arity.) A *semantic valuation* is a partial function from $\mathcal{X}$ to $\mathbf{D}$. Thus, $(\rho \in)$ $(\mathcal{X} \hookrightarrow \mathbf{D})$ is the set of semantic valuations. For a set I, $\vec{X} \in \mathcal{X}^I$, $\vec{p} \in \mathbf{D}^I$, let $\vec{S}/\vec{X}$ be the semantic valuation $\{(\vec{X}(i), \vec{p}(i)) : i \in I\}$, which maps $\vec{X}(i)$ to $\vec{p}(i)$ $(i \in I)$. For $S \in \Lambda(\mathcal{S}, \mathcal{X})$ and $\rho \in (\mathcal{X} \hookrightarrow \mathbf{D})$ with $\mathrm{Var}(S) \subseteq \mathrm{dom}(\rho)$, let $[\![S]\!]^J(\rho)$ denote the interpretation of S in J with the assignment of x to $\rho(x)$ $(x \in \mathrm{Var}(S))$; formally $[\![S]\!]^J(\rho)$ is defined by induction on the structure of S using the following rules: (i) $[\![X]\!]^J(\rho) = \rho(X)$, (ii) $[\![F(\langle S_i \rangle_{i \in r})]\!]^J(\rho) = J(F)(\langle [\![S_i]\!]^J(\rho) \rangle_{i \in r})$. Also, for $n \in \omega$, $\vec{S} \in (\Lambda(\mathcal{S}, \mathcal{X}))^n$, let $[\![\vec{S}]\!]^J(\rho) = \langle [\![\vec{S}(i)]\!]^J(\rho) \rangle_{i \in n}$. ■

The notion of *SOS format* is defined by:

Definition 10 Let $\langle X_n \rangle_{n \in \omega}, \langle Y_n \rangle_{n \in \omega} \in (\omega \to \mathcal{X})$ such that both $\langle X_n \rangle_{n \in \omega}$ and $\langle Y_n \rangle_{n \in \omega}$ are one-to-one and $\{X_n\}_{n \in \omega} \cap \{Y_n\}_{n \in \omega} = \emptyset$. For $r \in \omega$, let $\vec{X}_r = \langle X_0, \cdots, X_{r-1} \rangle$, and $\vec{Y}_r = \langle Y_0, \cdots, Y_{r-1} \rangle$.

(1) A *transition rule* (*TR*) is a pair of a subset $\{((S_i, \sigma_i), a_i, (S'_i, \sigma'_i))\}_{i \in I}$ of $(\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}) \times \Sigma) \times \mathbf{A} \times (\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X}) \times \Sigma)$ and an element $(S, (\sigma, a, \sigma'), S')$ of this set. A TR $R = (\{((S_i, \sigma_i), a_i, (S'_i, \sigma'_i))\}_{i \in I}, ((S, \sigma), a, (S', \sigma)))$ is usually written as:

$$\frac{\{(S_i, \sigma) \xrightarrow{a_i} (S'_i, \sigma)\}_{i \in I}}{(S, \sigma) \xrightarrow{a} (S', \sigma')}.$$

(2) A TR R is in the *SOS format* iff it is of the form:

$$\frac{\{(X_i, \sigma_i) \xrightarrow{a_i} (Y_i, \sigma'_i)\}_{i \in I}}{(F(\vec{X}_r), \sigma) \xrightarrow{a} (S, \sigma')}, \tag{2}$$

where $F \in \mathcal{F}^{(r)}$, $I \subseteq r$, and $S \in \Lambda(\mathcal{S}, \mathcal{X})$ with $\mathrm{Var}(S) \subseteq \{X_i : i \in r\} \cup \{Y_i : i \in I\}$.

(3) Let $(D \in)$ $\mathrm{Decl}(\mathcal{S}_{\mathrm{rec}}) = (\mathcal{Z} \to \wp(\Lambda(\mathcal{S}_{\mathrm{rec}}, \mathcal{X})^{\emptyset}))$. The elements of $\mathrm{Decl}(\mathcal{S}_{\mathrm{rec}})$ are called *declarations.* Let $D \in \mathrm{Decl}(\mathcal{S}_{\mathrm{rec}})$. The *recursion rules* under D is the TR's of the form:

$$\frac{\{(D(Z), \sigma) \xrightarrow{a} (s', \sigma')\}}{(Z, \sigma) \xrightarrow{a} (s', \sigma)},$$

where $Z \in \mathcal{Z}$. Let $\mathcal{R}_{\mathrm{rec}}(D)$ be the set of recursion rules under D.

(4) A *transition system specification* (TSS) is a 4-tuple $(\mathcal{S}_{\rm rec}, \Sigma, \mathbf{A}, \mathcal{R})$ with $\mathcal{S}_{\rm rec}$ a signature, Σ a set of states, $\mathbf{A}$ a set of actions, $\mathcal{R}$ a set of TR's. A TSS $\mathcal{T} = (\mathcal{S}_{\rm rec}, \Sigma, \mathbf{A}, \mathcal{R})$ is said to be in the *SOS format* iff there is a declaration D such that (3), (4) below hold:

$$\mathcal{R}_{\rm rec}(D) \subseteq \mathcal{R}; \tag{3}$$

all elements of $\mathcal{R} \setminus \mathcal{R}_{\rm rec}(D)$ are in the SOS format. ∎ (4)

Note that given a TSS $\mathcal{T} = (\mathcal{S}_{\rm rec}, \Sigma, \mathbf{A}, \mathcal{R})$ in the SOS format, a declaration D satisfying (3) and (4) above is uniquely determined; we denote such a declaration by $D_{\mathcal{T}}$. In the rest of this section, a TSS $\mathcal{T} = (\mathcal{S}_{\rm rec}, \Sigma, \mathbf{A}, \mathcal{R})$ in the SOS format is fixed, and let us use $\mathbf{S}$ to denote $\Lambda(\mathcal{S}_{\rm rec}, \mathcal{X})^{\emptyset}$.

Definition 11 Elements of $(\mathbf{S} \times \Sigma) \times \mathbf{A} \times (\mathbf{S} \times \Sigma)$ are called *transitions.* The notion of *provability* (or *derivability*) of *transitions* is defined in the same way as Definition 2.3 in [5]. Let $\rightarrow = \{((s, \sigma), a, (s', \sigma')) \in (\mathbf{S} \times \Sigma) \times \mathbf{A} \times (\mathbf{S} \times \Sigma) : ((s, \sigma), a, (s', \sigma'))$ is provable from $\mathcal{R}\}$.

Then, let $L(\mathcal{T})$, the *LTS defined by* $\mathcal{T}$, be defined by: $L(\mathcal{T}) = (\mathbf{S}, \mathbf{A}, \Sigma, \rightarrow)$. Let $\mathcal{O}^{\rm B}_{L(\mathcal{T})} : \mathbf{S} \rightarrow \mathbf{P}_{\rm B}$ be the semantic model defined in § 3.1 with $\mathcal{A} = L(\mathcal{T})$. ∎

Notation 3 In the sequel of this section, we sometimes write $\mathcal{O}$ for $\mathcal{O}^{\rm B}_{L(\mathcal{T})}$, for short. Also, for notational convenience in the sequel, let us put $\mathcal{O}[\![\vec{s}]\!] = \langle \mathcal{O}[\![\vec{s}(i)]\!] \rangle_{i \in n}$, for $n \in \omega$ and $\vec{s} \in \mathbf{S}^n$. ∎

The following lemma, which plays a key role in this section, shows how semantic operations can be derived from $\mathcal{T}$.

Lemma 7 **(1)** *Let $\mathcal{I}_{\rm B}(\mathcal{S})$ be the set of interpretations for the signature $\mathcal{S}$ with the domain $\mathbf{P}_{\rm B}$. For $F \in \mathcal{F}^{(r)}$ and $J \in \mathcal{I}_{\rm B}(\mathcal{S})$, one has $J(F) : (\mathbf{P}_{\rm B})^r \rightarrow \mathbf{P}_{\rm B}$ by definition. The set $\mathcal{I}_{\rm B}(\mathcal{S})$ is a complete metric space with the metric $d^{\rm B}_{\mathcal{I}}$ defined by: $d^{\rm B}_{\mathcal{I}}(J_1, J_2) = \sup\{d^{\rm B}_{F}(J_1(F), J_2(F)) : F \in \mathcal{F}\}$ $(J_1, J_2 \in \mathcal{I}_{\rm B}(\mathcal{S}))$, where for $F \in \mathcal{F}^{(r)}$, $d^{\rm B}_{F}$ is the metric on $((\mathbf{P}_{\rm B})^r \rightarrow \mathbf{P}_{\rm B})$ defined as in Lemma 1 (3). Let $\mathcal{I}^{\rm B}_{\rm NE}(\mathcal{S}) = \{J \in \mathcal{I}_{\rm B}(\mathcal{S}) : \forall F \in \mathcal{S}[\ J(F)$ is nonexpansive $]\}$. Then, $(\mathcal{I}^{\rm B}_{\rm NE}(\mathcal{S}), d^{\rm B}_{\mathcal{I}})$ is also a complete metric space.*

(2) *For a TR of the form (2) and $\vec{p} \in (\mathbf{P}_{\rm B})^r$, $\vec{q} \in (\mathbf{P}_{\rm B})^I$, let*

$$\rho_R(\vec{p}, \vec{q}) = \{(X_i, \vec{p}(i))\}_{i \in r} \cup \{(Y_i, \vec{q}(i))\}_{i \in I}.$$

By means of $\rho_R(\vec{p}, \vec{q})$, a mapping $\Psi_{\rm B} : \mathcal{I}^{\rm B}_{\rm NE}(\mathcal{S}) \rightarrow \mathcal{I}^{\rm B}_{\rm NE}(\mathcal{S})$ is defined as follows: For $J \in \mathcal{I}^{\rm B}_{\rm NE}(\mathcal{S})$, $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $\vec{p} \in (\mathbf{P}_{\rm B})^r$, let $\Psi_{\rm B}(J)(F)(\vec{p}) = (\widehat{\Psi}_{\rm B}(J)(F)(\vec{p}))^{\rm cls}$, where

$$\begin{aligned} &\widehat{\Psi}_{\rm B}(J)(F)(\vec{p}) = \\ &\{((\sigma, a, \sigma'), [\![S]\!]^J(\rho_R(\vec{p}, \vec{q}))) : \\ &\ R \in \mathcal{R} \wedge \\ &\ R = \frac{\{(X_i, \sigma_i) \xrightarrow{a_i} (Y_i, \sigma'_i)\}_{i \in I}}{(F(\vec{X}_r), \sigma) \xrightarrow{a} (S, \sigma')} \\ &\ \wedge \vec{q} \in (\mathbf{P}_{\rm B})^I \wedge \\ &\ \forall i \in I[\ ((\sigma_i, a_i, \sigma'_i), \vec{q}(i)) \in \vec{p}(i)\ ]\}. \end{aligned} \tag{5}$$

Then, the mapping $\Psi_{\rm B}$ is a contraction. ∎

Proof. The same as the Proof of Lemma 8 of [6], except that the action set $\mathbf{A}$ in [6] is replaced by $\Sigma \times \mathbf{A} \times \Sigma$. ∎

By means of $\Psi_{\rm B}$, an interpretation K of $\mathcal{S}$ based on the BT domain is defined by:

Definition 12 Let K be the fixed point of $\Psi_{\rm B}$. ∎

Then, one has the following lemma by definition.

Lemma 8 *For $F \in \mathcal{F}^{(r)}$ and $\vec{p} \in (\mathbf{P}_{\rm B})^r$, one has*

$$K(F)(\vec{p}) = (\widehat{\Psi}_{\rm B}(K)(F)(\vec{p}))^{\rm cls}, \tag{6}$$

where $\widehat{\Psi}_{\rm B}(K)(F)(\vec{p} \cdot \vec{\pi})$ is defined as in (5) with J replaced by K. Moreover, the mapping $K(F)$ is nonexpansive. ∎

Remark 1 It is easy to see that if $L(\mathcal{T})$ is image finite and the rules $R \in \mathcal{R}$ in SOS format satisfy the following condition $(*)$, then taking closure of right side of (6) can be omitted. $(*)$: *Given $F \in \mathcal{F}^{(r)}$, $\sigma \in \Sigma$, $a \in \mathbf{A}$, $\vec{s} \in \mathbf{S}^r$, there are only finite number of rules of the form (2) such that $\forall i \in I, \exists (s, \sigma)[\ (\vec{s}(i), \sigma_i) \xrightarrow{a_i} (s, \sigma)\ ]$.* ∎

Next, the denotational model $\mathcal{D}^{\rm B}_{\mathcal{T}}$ is defined compositionally from the interpretation K, with the meaning of each recursive statement as the fixed point of a certain higher-horder mapping:

Definition 13 A TSS $\mathcal{T} = (\mathcal{S}_{\rm rec}, \Sigma, \mathbf{A}, \mathcal{R})$ is saied to be in the Guarded SOS format (GSOS format) iff it is in the SOS format and there is $\mathcal{F}_{\rm g} \subseteq \mathcal{F}_{(0)}$ satisfying the following:

(i) For every $F \in \mathcal{F}_{\rm g}$, $K(F)$ is a contraction from $\mathbf{P}_{\rm F}$ to itself.

(ii) For every $Z, Z' \in \mathcal{Z}$, every occurrence Z' in $D_{\mathcal{T}}(Z)$ is in a subexpression of the form $F(\cdots)$ with some $F \in \mathcal{F}_{\rm g}$. ∎

Definition 14 Let $\Upsilon_{\rm B} : (\mathcal{Z} \rightarrow \mathbf{P}_{\rm B}) \rightarrow (\mathcal{Z} \rightarrow \mathbf{P}_{\rm B})$ be defined as follows: For every $H \in (\mathcal{Z} \rightarrow \mathbf{P}_{\rm B})$, $Z \in \mathcal{Z}$, $K \cup H$ is an interpretation of $\mathcal{S}_{\rm rec}$, and let $\Upsilon_{\rm B}(H)(Z) = [\![D(Z)]\!]^{K \cup H}$, where $[\![D(Z)]\!]^{K \cup H}$ is the interpretation of $D(Z) \in \Lambda(\mathcal{S}_{\rm rec}, \mathcal{X})^{\emptyset}$ in $K \cup H$. ∎

The following lemma follows immediately from the definition of GSOS format:

Lemma 9 *Let $\mathcal{T}$ be a TSS in the GSOS format. Then, $\Upsilon_{\rm B}$ is a contraction from $(\mathcal{Z} \rightarrow \mathbf{P}_{\rm B})$ to itself.* ∎

For the fixed point of $\Upsilon_{\rm B}$, the denotational BT model is defined by:

Definition 15 Let a TSS $\mathcal{T}$ in the GSOS format, and let $H(\mathcal{T}) = {\rm fix}(\Upsilon_{\rm B})$. We define $\mathcal{D}^{\rm B}_{\mathcal{T}} : \mathbf{S} \rightarrow \mathbf{P}_{\rm B}$ as follows: For every $s \in \mathbf{S}$, $\mathcal{D}^{\rm B}_{\mathcal{T}}[\![s]\!] = [\![s]\!]^{K \cup H(\mathcal{T})}$. ∎

3.3 Equivalence between $\mathcal{O}^{\rm B}_{L(\mathcal{T})}$ and $\mathcal{D}^{\rm B}_{\mathcal{T}}$

In this subsection, the semantic equivalence between $\mathcal{O}^{\rm B}_{L(\mathcal{T})}$ and $\mathcal{D}^{\rm B}_{\mathcal{T}}$ is established by showing that $\mathcal{O}^{\rm B}_{L(\mathcal{T})}$ is compositional in the sense of Lemma 10 below:

Lemma 10 *Let $\mathcal{T}$ be a TSS in the SOS format. Then,*

(1) *For $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $\vec{s} \in \mathbf{S}^r$, one has*

$$\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![F(\vec{s})]\!] = K(F)(\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![\vec{s}]\!]).$$

(2) *Let* $\tilde{H} = (\lambda Z \in \mathcal{Z} : \mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![Z]\!])$. *Then,*

$$\forall s \in \mathbf{S}[\ \mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![s]\!] = [\![s]\!]^{K\cup\tilde{H}}\ ].\blacksquare$$

Proof. The same as the proof of Theorem 1 of [6] with minor modifications to the nonuniform setting. ■

From Lemma 10 (2), the following lemma follows immediately.

Lemma 11 $\forall Z \in \mathcal{Z}[\ \mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![Z]\!] = \mathcal{D}^{\mathrm{B}}_{\mathcal{T}}[\![Z]\!]\ ]$. ■

Proof. From Lemma 10 (2), one has the mapping $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}\restriction\mathcal{Z} = (\lambda Z \in \mathcal{Z} : \mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![Z]\!])$ is also the fixed point of Υ_{B}. Thus, $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}\restriction\mathcal{Z} = \mathrm{fix}(\Upsilon_{\mathrm{B}}) = H(\mathcal{T})$. Consequently, for every $Z \in \mathcal{Z}$, one has $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![Z]\!] = H(\mathcal{T})(Z) = \mathcal{D}^{\mathrm{B}}_{\mathcal{T}}[\![A]\!]$. ■

From this lemma the semantic equivalence follows easily:

Theorem 1 *Let* $\mathcal{T}$ *be a TSS in the SOS format. Then,* $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})} = \mathcal{D}^{\mathrm{B}}_{\mathcal{T}}$. ■

Proof. By structural induction on $s \in \mathbf{S}$, one can obtain the following for every $s \in \mathbf{S}$,

$$\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![s]\!] = \mathcal{D}^{\mathrm{B}}_{\mathcal{T}}[\![s]\!]. \tag{7}$$

In the induction base, i.e., for $s \in \mathcal{Z}$, we obtain (7) immediately by Lemma 11; the induction step can be easily established by applying the compositionality of $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}$ (Lemma 10 (1)). ■

Thus, we have a denotational characterization of the operational model $\mathcal{O}^{\mathrm{B}}_{L(\mathcal{T})}[\![s]\!]$; For an advantage of having such a characterization, see [6] § 6.

4 Deriving Failures Models

In this section, we first define an operational failures model $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$ from a LTS $\mathcal{A}$, which is defined by a TSS $\mathcal{T}$. Then, a denotational failures model $\mathcal{D}^{\mathrm{F}}_{\mathcal{T}}$ is derived from $\mathcal{T}$, under the assumption that $\mathcal{T}$ fits into a more restricted format than the GSOS format; the format introduced in this section is based on the format due to De Simone [11] with certain additional restrictions specific to the nonuniform setting. Finally, the equivalence between $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$ and $\mathcal{D}^{\mathrm{F}}_{\mathcal{T}}$ is established.

A similar semantic equivalence problem for failures models was investigated by Vaandrager in the uniform setting [12]. However, the result of [12] in the uniform setting cannot be so directly extended to the nonuniform setting as in § 3. (See the Introduction, for the approach taken in the present paper.)

4.1 Deriving Operational Failures Model

First, an operational model for concurrency based on a variant $\mathbf{P}_{\mathrm{F}}$ of the *failures domain* [3] is induced by a LTS $\mathcal{A}$. First, the failures domain $\mathbf{P}_{\mathrm{F}}$ is defined. The domain $\mathbf{P}_{\mathrm{F}}$ is based on the original one [3], with certain modifications in accordance with the nonuniform setting.

Definition 16 (1) The set of *failures* , written ($q \in$) $\mathbf{Q}_{\mathrm{F}}$, is defined by:

$$\mathbf{Q}_{\mathrm{F}} = ((\Sigma\times\mathbf{A}\times\Sigma)^{<\omega}\cdot\mathbf{R}) \cup (\Sigma\times\mathbf{A}\times\Sigma)^{\omega},$$

where $\mathbf{R} = \{\langle(\sigma,\Gamma)\rangle : (\sigma,\Gamma) \in \Sigma\times\wp(\mathbf{A})\}$.

(2) For $q \in \mathbf{Q}_{\mathrm{F}}$, let $\mathrm{istate}(q) = \sigma$ if $q = \langle(\sigma,\Gamma)\rangle$ with some σ, Γ; otherwise $q = \langle(\sigma,a,\sigma')\rangle\cdot q'$ with some σ, a, σ', q', and let $\mathrm{istate}(q) = \sigma$. For $p \in \mathbf{P}_{\mathrm{F}}$ and σ, let $p\langle\sigma\rangle = \{q \in p : \mathrm{istate}(q) = \sigma\}$, and $\mathrm{act}(p,\sigma) = \{a \in \mathbf{A} : \exists\sigma'[\ p[\langle(\sigma,a,\sigma')\rangle] \neq \emptyset\ ]\}$.

(3) We say p satisfies the *disjointness inaction condition*, written $\mathrm{DIC}(p)$, iff $\forall r \in (\Sigma\times\mathbf{A}\times\Sigma)^{<\omega}[\ p[r] \neq \emptyset \Rightarrow \forall\sigma,\exists\mathcal{G} \in \wp(\wp(\mathrm{act}(p,\sigma)))[\ \mathcal{G} \neq \emptyset \wedge \forall\Gamma[\ \langle(\sigma,\Gamma)\rangle \in p[r] \Leftrightarrow \exists\Gamma' \in \mathcal{G}[\ \Gamma\cap\Gamma' = \emptyset\ ]]]]$.

(4) Let the domain of *failure sets*, written $\mathbf{P}_{\mathrm{F}}$, defined by: $\mathbf{P}_{\mathrm{F}} = \{p \in \wp(\mathbf{Q}_{\mathrm{F}}) : \mathrm{DIC}(p)\}$. ■

A metric d_{F} on $\mathbf{P}_{\mathrm{F}}$ can be defined in terms of *truncations* of sequences so that the following holds (cf. e.g. Lemma 15 of [7]):

Lemma 12 *The space* $(\mathbf{P}_{\mathrm{F}}, d_{\mathrm{F}})$ *is a complete metric space.* ■

On the basis of the domain $\mathbf{P}_{\mathrm{F}}$, the *operational failures model* $\mathcal{O}_{\mathrm{F}}$ is defined by:

Definition 17 For $(s,\sigma) \in \mathbf{S}\times\Sigma$, let $\mathrm{Act}(s,\sigma) = \{a \in \mathbf{A} : \exists(s',\sigma') \in \mathbf{S}\times\Sigma[\ (s,\sigma) \xrightarrow{a} (s',\sigma')\ ]\}$.

A mapping Φ_{F} from $(\mathbf{S}\to\mathbf{P}_{\mathrm{F}})$ to $(\mathbf{S}\to\mathbf{P}_{\mathrm{F}})$ is defined as follows: For $\mathcal{M} \in (\mathbf{S}\to\mathbf{P}_{\mathrm{F}})$, $s \in \mathbf{S}$,

$$\begin{aligned}&\Phi_{\mathrm{F}}(\mathcal{M})(s) = \\ &\{\ \langle(\sigma,\Gamma)\rangle : (\sigma,\Gamma) \in \Sigma\times\wp(\mathbf{A}) \\ &\quad \wedge\Gamma\cap\mathrm{Act}(s,\sigma) = \emptyset\} \\ &\cup(\bigcup\{\langle(\sigma,a,\sigma')\rangle\cdot\mathcal{M}(s') : \\ &\qquad (s,\sigma) \xrightarrow{a} (s',\sigma')\})^{\mathrm{cls}}.\blacksquare\end{aligned}$$

Lemma 13 *The function* Φ_{F} *is a contraction from* $(\mathbf{S}\to\mathbf{P}_{\mathrm{F}})$ *to* $(\mathbf{S}\to\mathbf{P}_{\mathrm{F}})$. ■

Thus, by Banach's fixed point theorem, Φ_{F} has a unique fixed point, which is defined to be the operational failures model:

Definition 18 Let $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$ be the unique fixed point of Φ_{F}. ■

Then the following holds by the definition of $\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}$:

Lemma 14 *For every* $s \in \mathbf{S}$,

$$\begin{aligned}&\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}[\![s]\!] = \\ &\{\langle(\sigma,\Gamma)\rangle \in \mathbf{R} : \Gamma\cap\mathrm{Act}(s,\sigma) = \emptyset\}\cup \\ &(\bigcup\{\langle(\sigma,a,\sigma')\rangle\cdot\mathcal{O}^{\mathrm{F}}_{\mathcal{A}}[\![s']\!] : (s,\sigma) \xrightarrow{a} (s',\sigma')\})^{\mathrm{cls}}.\blacksquare\end{aligned}$$

4.2 Deriving Denotational Failures Model

In this subsection, it is shown that denotational models for concurrency based on the failures domain can be derived from the TSS $\mathcal{T}$, when the TSS fits into to a more restricted format, which is based the format due to De Simone [11].

Definition 19 (1) Let $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $I \subseteq r$, $\vec{g} : (\Sigma \to \Sigma)^r$, and $g' : \Sigma \times \Sigma^I \to \Sigma$. A TR R is a rule for F in the *State-Restricted Copy-Free SOS format* (*SRCF-SOS format*) w.r.t. I, $\vec{g}$ and g' iff it is of the form:

$$\frac{\{(X_i, \vec{g}(i)(\sigma)) \xrightarrow{a_i} (Y_i, \sigma_i')\}_{i \in I}}{(F(\vec{X}_r), \sigma) \xrightarrow{a} (S, g'(\sigma, \langle \sigma_i' \rangle_{i \in I}))}, \quad (8)$$

where $S \in \Lambda^1(\mathcal{S}, \mathcal{X})$ with $\mathrm{Var}(S) \subseteq \{X_i : i \in r \setminus I\} \cup \{Y_i : i \in I\}$. Let $\mathcal{R}(r, F, I, \vec{g}, g')$ be the set of these rules.

(2) A TSS $\mathcal{T} = (\mathcal{S}, \Sigma, \mathbf{A}, \mathcal{R})$ is said to be in the *SRCF-SOS format* iff there are mappings $\langle \vec{g}_F \rangle_{F \in \mathcal{F}}$ and $\langle g_F' \rangle_{F \in \mathcal{F}}$ such that (9), (10), (11) below hold:

$$\forall r \in \omega, \forall F \in \mathcal{F}^{(r)} [\ \vec{g}_F \in (\Sigma \to \Sigma)^r \wedge g_F' \in \textstyle\prod_{I \in \wp(r)} [\Sigma \times \Sigma^I \to \Sigma]\ ]; \quad (9)$$

$$\mathcal{R}_{\mathrm{rec}}(D) \subseteq \mathcal{R}, \quad (10)$$

where $\mathcal{R}_{\mathrm{rec}}(D)$ is the set of recursion rules defined in Definition 10;

$$\mathcal{R} \setminus \mathcal{R}_{\mathrm{rec}}(D) \subseteq \bigcup \{\mathcal{R}(r, F, I, \vec{g}_F, g_F'(I)) : r \in \omega \wedge F \in \mathcal{F}^{(r)} \wedge I \subseteq r\ \}. \quad (11)$$ ∎

In the rest of this section, a TSS $\mathcal{T} = (\mathcal{S}, \Sigma, \mathbf{A}, \mathcal{R})$ in the SRCF-SOS format is fixed with $\langle \vec{g}_F \rangle_{F \in \mathcal{F}}$ and $\langle g_F' \rangle_{F \in \mathcal{F}}$ satisfying (9).

The following definition is given as a preliminary to the definition of semantic operation:

Definition 20 (1) Let ref, apart : $\wp(\mathbf{Q}_{\mathrm{F}}) \to \mathbf{R}$ be defined as follows: For every $p \in \wp(\mathbf{Q}_{\mathrm{F}})$, $\mathrm{ref}(p) = p \cap \mathbf{R}$, and $\mathrm{apart}(p) = p \setminus \mathrm{ref}(p)$.

(2) Let $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $\sigma \in \Sigma$, and $a \in \mathbf{A}$. First, let $\mathcal{H}_F(\sigma, a) \subseteq (\wp(\mathbf{A}))^r$ be defined by:

$$\mathcal{H}_F(\sigma, a) = \{\vec{\Gamma} \in (\wp(\mathbf{A}))^r : \exists R \in \mathcal{R}[\ R = \frac{\{(X_i, \vec{g}_F(i)(\sigma)) \xrightarrow{a_i} (Y_i, \sigma_i')\}_{i \in I}}{(F(\vec{X}), \sigma) \xrightarrow{a} (S, g_F'(I)(\sigma, \vec{\sigma}'))} \wedge \forall i \in I[\ a_i \in \vec{\Gamma}(i)\ ]]\},$$

where $\vec{\sigma}' = \langle \sigma_i' \rangle_{i \in I}$. We set $\mathcal{H}_F^{\mathrm{c}}(\sigma, a) = (\wp(\mathbf{A}))^r \setminus \mathcal{H}_F(\sigma, a)$. For $\Delta \subseteq (\wp(\mathbf{A}))^r$, let $\Delta^{\ominus} = \{\langle \mathbf{A} \setminus \vec{\Gamma}(i) \rangle_{i \in r} : \vec{\Gamma} \in \Delta\}$.

(3) Let us define $\mathrm{rec}_\sigma^F : (\wp(\wp(\mathbf{A})))^r \to \wp(\wp(\mathbf{A}))$ as follows: For $\vec{\mathcal{G}} \in (\wp(\wp(\mathbf{A})))^r$,

$$\mathrm{rec}_\sigma^F(\vec{\mathcal{G}}) = \{\Gamma : \textstyle\prod_{i \in r} [\vec{\mathcal{G}}(i)] \cap (\bigcap_{a \in \mathbf{A}} [\mathcal{H}_F^{\mathrm{c}}(\sigma, a)])^{\ominus} \neq \emptyset\ \}.$$

Then, let

$$\mathrm{rec}^F(\vec{\mathcal{G}}) = \bigcup_{\sigma \in \Sigma} [\mathrm{rec}_\sigma^F(\vec{\mathcal{G}})].$$ ∎

Lemma 15 (1) *Let $\mathcal{I}_{\mathrm{F}}(\mathcal{S})$ be the set of interpretations for the signature $\mathcal{S}$ with the domain $\mathbf{P}_{\mathrm{F}}$. For $F \in \mathcal{F}^{(r)}$ and $J \in \mathcal{I}_{\mathrm{F}}(\mathcal{S})$, one has $J(F) : (\mathbf{P}_{\mathrm{F}})^r \to \mathbf{P}_{\mathrm{F}}$ by definition. The set $\mathcal{I}_{\mathrm{F}}(\mathcal{S})$ is a complete metric space with the metric $d_{\mathcal{I}}^{\mathrm{F}}$ defined by: $d_{\mathcal{I}}^{\mathrm{F}}(J_1, J_2) = \sup\{d_F^{\mathrm{F}}(J_1(F), J_2(F)) : F \in \mathcal{S}\}$ ($J_1, J_2 \in \mathcal{I}_{\mathrm{F}}(\mathcal{S})$), where for $F \in \mathcal{F}^{(r)}$, d_F^{F} is the metric on $((\mathbf{P}_{\mathrm{F}})^r \to \mathbf{P}_{\mathrm{F}})$ defined as in Lemma 1 (3). Let $\mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S}) = \{J \in \mathcal{I}_{\mathrm{F}}(\mathcal{S}) : \forall F \in \mathcal{S}[\ J(F)$ is nonexpansive $]\}$. Then, $(\mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S}), d_{\mathcal{I}}^{\mathrm{F}})$ is also a complete metric space.*

(2) *For a TR R of the form (8) and $\vec{p} \in (\mathbf{P}_{\mathrm{F}})^r$, let*

$$\rho_R(\vec{p}) = \{(Y_i, \vec{p}(i)[\langle (\sigma_i, a_i, \sigma_i') \rangle])\}_{i \in I} \cup \{(\vec{X}(i), \vec{p}(i)) : i \in r \setminus I\}.$$

Then, for $J \in \mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S})$, let

$$\psi_R(J, \vec{p}) = \langle (\sigma, a, \sigma') \rangle \cdot [\![S]\!]^J(\rho_R(\vec{p})).$$

By means of $\psi_R(J, \vec{p})$, a mapping $\Psi_{\mathrm{F}} : \mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S}) \to \mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S})$ is defined as follows: For $J \in \mathcal{I}_{\mathrm{NE}}^{\mathrm{F}}(\mathcal{S})$, $F \in \mathcal{F}^{(r)}$, $\vec{p} \in (\mathbf{P}_{\mathrm{F}})^r$ let $\Psi_{\mathrm{F}}(J)(F)(\vec{p}) = \mathrm{ref}_{\mathrm{F}}(\langle \mathrm{ref}(\vec{p}(i)) \rangle_{i \in r}) \cup (\widehat{\Psi}_{\mathrm{F}}(J)(F)(\vec{p}))^{\mathrm{cls}}$, where

$$\widehat{\Psi}_{\mathrm{F}}(J)(F)(\vec{p}) = \bigcup \{\psi_R(J, \vec{p}) : R \in \mathcal{R} \wedge R = \frac{\{(X_i, \sigma_i) \xrightarrow{a_i} (Y_i, \sigma_i')\}_{i \in I}}{(F(\vec{X}_r), \sigma) \xrightarrow{a} (S, \sigma')} \wedge \forall i \in I[\ \vec{p}(i)[\langle (\sigma_i, a_i, \sigma_i') \rangle] \neq \emptyset\ ]\}. \quad (12)$$

Then, the mapping Ψ_{F} is a contraction. ∎

By means of Ψ_{F}, an interpretation M of $\mathcal{S}$ based on the failures domain is defined by:

Definition 21 Let M be the fixed point of Ψ_{F}. ∎

Then, one has the following lemma by definition:

Lemma 16 *For $r \in \omega$, and $F \in \mathcal{F}^{(r)}$, $\vec{p} \in (\mathbf{P}_{\mathrm{F}})^r$, one has*

$$M(F)(\vec{p}) = \mathrm{ref}_{\mathrm{F}}(\langle \mathrm{ref}(\vec{p}(i)) \rangle_{i \in r}) \cup (\widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p}))^{\mathrm{cls}}, \quad (13)$$

where $\widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p})$ is defined as in (12) with J replaced by M. Moreover, the mapping $M(F)$ is nonexpansive. ∎

For a TSS $\mathcal{T}$ in the GSOS format, the *denotational failures model* $\mathcal{D}_{\mathcal{T}}^{\mathrm{F}}$ is defined as the denotational BT model $\mathcal{D}_{\mathcal{T}}^{\mathrm{B}}$ has been defined in § 3.2, but using $\mathbf{P}_{\mathrm{F}}$ and M instead of $\mathbf{P}_{\mathrm{B}}$ and K, respectively.

4.3 Equivalence between $\mathcal{O}_{L(\mathcal{T})}^{\mathrm{F}}$ and $\mathcal{D}_{\mathcal{T}}^{\mathrm{F}}$

When a TSS $\mathcal{T}$ is in the SRCF-SOS format with $L(\mathcal{T})$ being image finite, the interpretation $M(F)$ of every function symbol $F \in \mathcal{F}$ is *distributive* in the sense of Lemma 17 below; from the distributivity, the compositionality of $\mathcal{O}_{L(\mathcal{A})}^{\mathrm{F}}$, and therefore, the equivalence between $\mathcal{O}_{L(\mathcal{A})}^{\mathrm{F}}$ and $\mathcal{D}_{L(\mathcal{A})}^{\mathrm{F}}$ follows easily. It is for ensuring the distributivity that several additional restrictions are imposed in Definition 19 (1), on top of the restrictions in Definition 10 (2).

Lemma 17 *Let $\mathcal{T}$ be a TSS in the SRCF-SOS format such that $L(\mathcal{T})$ is image finite. Then,*

(1)
$$\forall i \in r, \forall \vec{p} \in (r \setminus \{i\} \to \mathbf{P}_{\mathrm{F}}), \forall p^{(0)}, p^{(1)} \in \mathbf{P}_{\mathrm{F}}[\ K(F)(\vec{p} \cup \{(i, p^{(0)} \cup p^{(1)})\}) = \textstyle\bigcup_{j \in 2} [K(F)(\vec{p} \cup \{(i, p^{(j)})\})]\ ].$$

(2) *Let $S \in \Lambda^1(\mathcal{S}, \mathcal{X})$, $\mathcal{Y} \in \wp_{\mathrm{f}}(\mathcal{X})$, $X \in \mathcal{X}$ such that $\mathrm{Var}(S) \subseteq \mathcal{Y} \cup \{X\}$. Then,*

$$\forall \rho \in (\mathcal{Y} \to \mathbf{P}_{\mathrm{F}}), \forall p^{(0)}, p^{(1)} \in \mathbf{P}_{\mathrm{F}}[\ [\![S]\!]^M(\rho \cup \{(X, p^{(0)} \cup p^{(0)})\}) = \textstyle\bigcup_{j \in 2} [[\![S]\!]^M(\rho \cup \{(X, p^{(j)})\})]\ ].$$ ∎

Lemma 18 *Let $\mathcal{T}$ be a TSS in the SRCF-SOS format such that $L(\mathcal{T})$ is image finite. Then,*

(1) *For $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $\vec{s} \in \mathbf{S}^r$, one has*
$$\mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}[\![F(\vec{s})]\!] = M(F)(\mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}[\![\vec{s}]\!]).$$

(2) *Let $\widehat{H} = (\lambda Z \in \mathcal{Z} : \mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}[\![Z]\!])$. Then,*
$$\forall s \in \mathbf{S}[\ \mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}[\![s]\!] = [\![S]\!]^{M \cup \widehat{H}}\ ].\blacksquare$$

Proof. We sketch the proof of Part (1); Part (2) follows immediately from Part (1). First, we give a few notational preliminaries: For $n \in \omega$, $p_1, p_2 \in \mathbf{P}_{\mathrm{F}}$, we write $p_1 \simeq_n p_2$ to denote that $d_{\mathrm{F}}(p_1, p_2) \leq (1/2)^n$; we write $\mathcal{O}$ to denote $\mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})}$, for simplicity. Further, for $n \in \omega$, let us use $P(n)$ and $P'(n)$ to denote the following propositions (14) and (15), respectively.

$$P(n) : \forall r \in \omega, \forall F \in \mathcal{F}^{(r)}, \vec{s} \in \mathbf{S}^r[\ \mathcal{O}[\![F(\vec{s})]\!] \simeq_n M(F)(\mathcal{O}[\![\vec{s}]\!])\ ]. \tag{14}$$

$$P'(n) : \forall S \in \Lambda(\mathcal{S}, \mathcal{X}), \forall \zeta \in (\mathcal{X} \hookrightarrow \mathbf{S})[\ \mathrm{Var}(S) \subseteq \mathrm{dom}(\zeta) \Rightarrow \mathcal{O}[\![S[\zeta]]\!] \simeq_n [\![S]\!]^M(\mathcal{O} \circ \zeta)\ ]. \tag{15}$$

Then, we observe that $(*)$: $\forall n \in \omega[\ P(n) \Rightarrow P'(n)\ ]$. We can show by induction on n that $\forall n \in \omega[\ P(n)\ ]$, which implies the claim of Part (1). One has $P(0)$ immediately; supposing $P(n)$, let us show $P(n+1)$. One has $P'(n)$ by $(*)$. Let $r \in \omega$, $F \in \mathcal{F}^{(r)}$, $\vec{s} \in \mathbf{S}^r$. Putting $\vec{p} = \mathcal{O}[\![\vec{s}]\!]$, let us show $(\dagger)$: $\mathcal{O}[\![F(\vec{s})]\!] \simeq_{n+1} M(F)(\vec{p})$. Since $\mathcal{O}[\![F(\vec{s})]\!] = \mathrm{ref}(\mathcal{O}[\![F(\vec{s})]\!]) \cup \mathrm{apart}(\mathcal{O}[\![F(\vec{s})]\!])$ and $M(F)(\vec{p}) = \mathrm{ref}_{\mathrm{F}}(\langle \mathrm{ref}(\vec{p}(i)) \rangle_{i \in r}) \cup \widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p})$, it suffices to show $(\ddagger)$: $\mathrm{ref}(\mathcal{O}[\![F(\vec{s})]\!]) \simeq_{n+1} \mathrm{ref}_{\mathrm{F}}(\langle \mathrm{ref}(\vec{p}(i)) \rangle_{i \in r})$, and $(**)$: $\mathrm{apart}(\mathcal{O}[\![F(\vec{s})]\!]) \simeq_{n+1} \widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p})$. One obtains $(\ddagger)$ easily from the definition of $\mathrm{ref}_{\mathrm{F}}$; let us show $(**)$. For $R \in \mathcal{R}$ of the form:

$$\frac{\{(X_i, \vec{g}_F(i)(\sigma)) \xrightarrow{a_i} (Y_i, \sigma'_i)\}_{i \in I}}{(F(\vec{X}), \sigma) \xrightarrow{a} (S, g'_F(I)(\sigma, \vec{\sigma}')),} \tag{16}$$

with $\vec{\sigma}' = \langle \sigma'_i \rangle_{i \in I}$, and for $\vec{t} \in \mathbf{S}^I$, let $\zeta_R(\vec{s}, \vec{t}) = \{(X_i, \vec{s}(i))\}_{i \in r} \cup \{(Y_i, \vec{t}(i))\}_{i \in I}$. Then,

$$\begin{aligned}
&\mathrm{apart}(\mathcal{O}[\![F(\vec{s})]\!]) \\
&= \bigcup\{\langle(\sigma, a, g'_F(\sigma, \vec{\sigma}'))\rangle \cdot \mathcal{O}[\![S[\zeta_R(\vec{s}, \vec{t})]]\!] : \\
&\qquad R \in \mathcal{R} \wedge R \text{ is of the form (16)} \wedge \\
&\qquad \vec{t} \in \mathbf{S}^I \wedge \forall i \in I[\ (\vec{s}(i), \vec{g}_F(i)(\sigma)) \xrightarrow{a_i} (\vec{t}(i), g'_F(I)(\sigma, \vec{\sigma}'))\ ]\} \\
&\simeq_{n+1} \bigcup\{\langle(\sigma, a, g'_F(\sigma, \vec{\sigma}'))\rangle \cdot [\![S]\!]^M(\mathcal{O} \circ \zeta_R(\vec{s}, \vec{t})) : \\
&\qquad \text{the same condition for } R, \vec{t} \text{ as above}\ \} \\
&\qquad (\text{since } P'(n)) \\
&= \bigcup\{\langle(\sigma, a, g'_F(\sigma, \vec{\sigma}'))\rangle \cdot [\![S]\!]^M(\rho_R(\vec{p})) : \\
&\qquad R \in \mathcal{R} \wedge R \text{ is of the form (16)} \wedge \\
&\qquad \forall i \in I[\ \vec{p}(i)[\langle(\sigma_i, a_i, \sigma'_i)\rangle] \neq \emptyset\ ]\} \\
&\qquad (\text{by applying Lemma 17 (2) several times, recalling } \rho_R(\vec{p}) \text{ is an expression introduced in Lemma 15 (2)}) \\
&= \widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p}) \\
&\qquad (\text{by the definition of } \widehat{\Psi}_{\mathrm{F}}(M)(F)(\vec{p}) \text{ introduced in Lemma 15 (2)})
\end{aligned}$$

Thus, one has $(**)$. $\blacksquare$

From Lemma 18, the following theorem follows immediately, just as Theorem 1 follows from Lemma 10:

Theorem 2 *Let $\mathcal{T}$ be a TSS in the SRCF-SOS format such that such that $L(\mathcal{T})$ is image finite. Then, one has $\mathcal{O}^{\mathrm{F}}_{L(\mathcal{T})} = \mathcal{D}^{\mathrm{F}}_{\mathcal{T}}$.* $\blacksquare$

References

[1] J.W. de Bakker, J.I. Zucker (1982), Processes and the denotational semantics of concurrency, *Inform. and Control* 54, pp. 70–120.

[2] B. Bloom, S. Istrail, A.R. Meyer (1988), Bisimulation can't be traced: preliminary report, in *Proceedings of the 15th POPL*, San Diego, California, 1988, pp. 229–239.

[3] S.D. Brookes, C.A.R. Hoare, and A.W. Roscoe (1984). A theory of communicating sequential processes, *Journal of the Association for Computer Machinery, Vol.31*, pp. 560–599.

[4] J. Dugundji (1966), *Topology*, Allyn and Bacon, Boston.

[5] J.F. Groote, F. Vaandrager (1988), *Structured operational semantics and bisimulation as a congruence*, Technical Report CS-R8845, Centre for Mathematics and Computer Science, Amsterdam. (To appear in Information and Computation. Extended abstract in *Proceedings 16th ICALP, Stresa, Lecture Notes in Computer Science, Vol.372*, pp. 423–438, Springer.)

[6] E. Horita (1992), Deriving compositional models for concurrency based on de Bakker-Zucker metric domain from structured operational semantics, *IEICE Transactions on Information and Systems Vol. E75-A, No. 3*, pp. 400–409.

[7] E. Horita, J.W. de Bakker, and J.J.M.M. Rutten (1990), *Fully abstract denotational models for nonuniform concurrent languages, CWI Report CS-R9027*, Amsterdam. (To appear in Information and Computation.)

[8] R. Milner (1980), *"A Calculus of Communicating Systems"* Lecture Notes in Computer Science, Vol. 92, Springer.

[9] G.D. Plotkin (1981), *A structural approach to operational semantics*, Report DAIMI FN-19, Comp. Sci. Dept., Aarhus Univ.

[10] J.J.M.M. Rutten (1990), Deriving denotational models for bisimulation from Structured Operational Semantics, in *Proceedings of IFIP TC2 Working Conference on Programming Concepts and Methods* (M. Broy, C.B. Jones, eds.), April 2–5, 1990, Sea Gallilee, Israel, pp. 155–178, North-Holland.

[11] R. de Simone (1985), Higher-level synchronising devices in MEIJE-SCCS, *Theoretical Computer Science* 37, pp. 245–267.

[12] F.W. Vaandrager (1991), On the relationship between process algebra and input/output automata (extended abstract), in *Proceedings of LICS'91*, IEEE Computer Sciety Press, pp. 387–398. A draft full version was communicated privately in June, 1992.