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立ち上がり時間のある制限式交替待ち行列

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ポアソン到着の待ち行列を二つ持つ処理系を考える。処理の方法を単一制限式であるとする。更にシステムの立ち上がり時間を考慮に入れる。実際のシュミレーションで、立ち上がり時間がスイッチオーバー時間に比べて大きく、無視出来ない時に有効な考察となる。我々の結果はＢｏｘｍａとＣｏｈｅｎによる立ち上がり時間を無視した場合の考察を拡張したものである。各待ち行列の母関数は解析関数であるので、問題をＲｉｅｍａｎｎ－Ｈｉｌｂｅｒｔ型境界値問題の形に定式化して母関数を決定し、仕事の平均待ち時間を、Ｃａｕｃｈｙ積分と等角写像を用いて計算する。立ち上がり時間が二つの待ち行列で等しいときは、比較的数値計算が容易である。

ALTERNATING SERVICE QUEUES WITH SETUP TIMES

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A single server system with two Poissonian arrival queues is considered. Each queue is served in 1-limited way with 'setup time'. This system is useful for the actual simulation when the set-up time, much longer than switching time, is not negligible.

Our result is an extension of those of Boxma and Cohen for the case without setup time. The generating function of the length of each queue is an analytic function. Therefore, formulating our problem as a Riemann-Hilbelt boundary value problem, we determine the generating functions and give a formula to obtain the waiting time of customers by means of a conformal mapping and Cauchy integrals. The hardness of its numerical evaluation depends on whether the setup time of a queue is the same as that of another queue.

Alternating service queues with setup times.

Introduction.

Recently, Cohen and Boxma has been conducting a series of analytic studies for systems involving two queues using a technique of so-called *Riemann-Hilbert type boundary value problem* . Some of them deal with *alternating service queues* such that two queues with independent Poisson arrival processes are attended by a single server alternately. With respect to the number of messages the server continues to serve at each visit to a queue, we may think of four disciplines: exhaustive, gated, limited, and decrementing (see Takagi [12] for terminology). When one of the queues has exhaustive service discipline or when both queues have gated service disciplines, we do not need the Riemann-Hilbert type formulation. Therefore, our interest here falls in the case where the two queues have both limited, both decrementing, or a mixture of limited and decrementing service disciplines. (We have no results for a mixture of the gated service and the limited or decrementing service disciplines.)

Another criterion for classification is whether a nonzero time is required for the server to switch from one queue to another. Finally, there is a case where the two queues are statistically symmetric (both have the same parameters), and a case otherwise. According to the three criteria mentioned above, previous works on this subject are classified as follows. For limited service systems without switchover times, Cohen and Boxma [6] consider symmetric models, while Eisenberg [8] and Cohen and Boxma [7](sec.III.2 and IV.1) deal with asymmetric models. For cases with switchover times, symmetric limited service systems are analyzed by Iisaku et al. [10] and Boxma [1], and asymmetric limited service systems are treated by Boxma and Groenendijk [3]. An asymmetric, decrementing service is discussed by Cohen [5]. Finally, Coffman et al. [4] analyze a system of two queues with alternating service periods, where an upper bound on the service period duration is posed at each queue.

A feature common to these works is that they require the reader a background of mathematics, particularly, that of complex analysis. Another point is that some of them provide only formal solution without rendering themselves to numerical evaluation procedures. Consequently, useful results in the above-cited papers are seldom utilized by practically-minded people. The purpose of this paper is to explain the course of analysis and the typical technique, considering a system of limited queues with set-up times. This system has applications in computer communication systems (e.g., half duplex transmission).

The rest of the paper is organized as follows. In section 1, we prepare notation and formulate the problem. In section 2, we demonstrate how to translate the problem into a Riemann-Hilbert boundary value problem using an easy "kernel" (a function of two variables which appears in the denominator of the governing equation) to clarify the course of analysis. In section 3, we show technique to analyze the zeroes of the original kernel on the complex plane. Finally, we comment on the numerical evaluation of the waiting time in section 4.

1. Formulation

We consider a system consisting of a single server with two Poisson arrival streams of customers with arrival rate λ_1 and λ_2.

The service is done in 1-limited manner, which means that the server leaves a queue after serving just a customer of the queue (if it is not vacant). Besides, we consider the "set-up time" of the system. If both queues become vacant, the system sleeps until a customer comes in. If a customer arrives at queue i during a sleeping state, the system wakes up after some set-up time S_i ($i = 1, 2$) . $S_i^*(s)$ is the LST (Laplace-Stieltjes Transform) of the distribution function of the set-up time. $s_i = E(S_i)$ and $s_i^{(2)} = E(S_i^2)$ are the mean and the second moment of S_i.

$B_i^*(s)$ is the LST of the distribution function of the service time B_i for a customer of type i ($i = 1, 2$). $b_i = E(B_i)$ and $b_i^{(2)} = E(B_i^2)$ are the mean and the second moment of B_i. The total server utilization is $\rho = \lambda_1 b_1 + \lambda_2 b_2$. Then $B_i^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2)$ is the generating function of the number of the customers flowing into the system while a type-i customer is being served, where $\lambda = \lambda_1 + \lambda_2$. Let $r_i = \dfrac{\lambda_i}{\lambda}$ for $i = 1, 2$.

Let us take the departure epochs of messages from both queues as Markov points. We assume the process is stationary.

We define the generating function of the number of customers of the queues at the departure epoch from queue i by

$$Q_i(z_1, z_2) \overset{\text{def}}{=} \sum_{j,k=0}^{\infty} p_i(j,k)\, z_1^j\, z_2^k,$$

where

$p_i(j,k) = Prob\{$ length (Queue 1) $= j$ and length (Queue 2) $= k$; at the departure epoch from the queue i $\}$.

For convenience, we write P_0 for $Q_1(0, 0) + Q_2(0, 0)$, which is given by

$$P_0 = \frac{1-\rho}{1 + \lambda_1 s_1 + \lambda_2 s_2} \qquad \text{(See [13]).} \tag{1.1}$$

Then the following formula holds:

$$Q_1(z_1, z_2) = \frac{B_1^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2)}{z_1}\{Q_2(z_1, z_2) - Q_2(0, z_2) + Q_1(z_1, 0) - Q_1(0, 0)\} + P_0 r_1 B_1^* S_1^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2). \tag{1.2}$$

From symmetry of the system, we have

$$Q_2(z_1, z_2) = \frac{B_2^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2)}{z_2}\{Q_1(z_1, z_2) - Q_1(z_1, 0) + Q_2(0, z_2) - Q_2(0, 0)\} + P_0 r_2 B_2^* S_2^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2). \tag{1.3}$$

Combining (1.2) and (1.3), we have the following:

$$2Q_1(z_1, z_2)\{z_1z_2 - B_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2)B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\} = \{z_2 - B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\}B_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2)$$

$$[2Q_1(z_1, 0) - Q_1(0,0) - 2Q_2(0, z_2) + Q_2(0,0) + \{z_1r_1S_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2) - z_2r_2S_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\}P_0 -$$

$$\{\frac{z_2 + B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)}{z_2 - B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)}\}\{1 - z_1r_1S_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2) - z_2r_2S_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\}P_0]. \tag{1.4}$$

The mean waiting time $E(W_1)$ of type 1 customer is given by a standard M/G/1-type argument. We have:

$$E(W_1) = \frac{1}{\lambda_1}\{\frac{d}{dz}Q_1(1,z)\}_{z=1} - b_1. \tag{1.5}$$

The derivative follows (1.4) after a straightforward calculation.

$$\frac{dQ_1(z,1)}{dz}\ |_{z=1} = 2(\frac{-\lambda_1b_2}{1 - \lambda_1b_1 - \lambda_1b_2})\frac{dQ_1(z,0)}{dz}\ |_{z=1} + P_0J, . \tag{1.6}$$

where

$$J = \frac{\lambda_1b_2}{1 - \lambda_1b_1 - \lambda_1b_2}\{2r_1\lambda_1s_1 + \frac{b_2(2r_1s_1 + r_1\lambda_1s_1^{(2)} + r_2\lambda_1s_2^{(2)}) - b_2^{(2)}(r_1 + r_1\lambda_1s_1 + r_2\lambda_1s_2)}{b_2^2.}\}$$

$$-\{\frac{\lambda_1^2b_2^{(2)} + \lambda_1^2b_1b_2}{2(1 - \lambda_1b_1 - \lambda_1b_2)} + \frac{\lambda_1^3b_2(b_1^{(2)} + b_2^{(2)} + 2b_1b_2)}{2(1 - \lambda_1b_1 - \lambda_1b_2)^2}\}\{r_1 - r_2 - \frac{2(r_1 + r_1\lambda_1s_1 + r_2\lambda_1s_2)}{\lambda_1b_2}\}. \tag{1.7}$$

Thus in order to obtain the mean waiting time, we must analyze the function $Q_1(z,0)$ and get its derivative at $z = 1$.

Rewriting (1.4) by using the notation

$$\sigma_1(z_1) \stackrel{\text{def}}{=} 2Q_1(z_1,0) - Q_1(0,0), \tag{1.8.1}$$

$$\sigma_2(z_2) \stackrel{\text{def}}{=} 2Q_2(0, z_2) - Q_2(0,0), \tag{1.8.2}$$

and

$$G(\lambda_1z_1 + \lambda_2z_2) \stackrel{\text{def}}{=} -\{z_1r_1S_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2) - z_2r_2S_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)$$

$$-\frac{z_2 + B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)}{z_2 - B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)}(1 - z_1r_1S_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2) - z_2r_2S_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2))\}P_0, \tag{1.8.3}$$

we have the following:

$$2Q_1(z_1, z_2) = \frac{B_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\{z_2 - B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)\}\{\sigma_1(z_1) - \sigma_2(z_2) - G(\lambda_1z_1 + \lambda_2z_2)\}}{z_1z_2 - B_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2)B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)} \tag{1.9}$$

2. Boundary value problem with an easy kernel

We must consider the variety V consisting of zeros of the "kernel" $z_1z_2 - B_1^*(\lambda - \lambda_1z_1 - \lambda_2z_2)B_2^*(\lambda - \lambda_1z_1 - \lambda_2z_2)$ of (1.9) in order to obtain the derivative $\frac{dQ_1(z,0)}{dz}\ |_{z=1}$. However, before that, in order to show the principle of analysis, we shall demonstrate how we formulate this type of problem into a boundary value problem and solve it. To do so, we use an easy example when the kernel is $z_1z_2 - 1$ (although it does not represent any system).

The zero of the kernel is $V_0 = \{(z_1, z_2):\ z_1z_2 - 1 = 0\}$. In order to simplify the analysis, we also assume here that $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$.

Suppose (z_1, z_2) is an element of V_0. Equation (1.9) implies that

$$\sigma_1(z_1) - \sigma_2(z_2) = G(\lambda_1 z_1 + \lambda_2 z_2) \tag{2.1}$$

on V_0 provided that $|z_i| \leq 1$ for $i = 1,2$.
Since $\lambda_1 = \lambda_2$, and since $|z_1| \leq 1$ and $|z_2| \leq 1$ simultaneously hold on V_0 if and only if $|z_1| = 1$, we obtain

$$\sigma_1(z) - \sigma_2(z^{-1}) = G(\lambda \;\; Re(z)) \tag{2.2}$$

on the unit circle $C = \{z \colon |z| = 1\}$.

Now we have the following:

Observation 2.3.

1. $\sigma_1(z)$ is regular on the unit disk $C^+ = \{z \colon |z| < 1\}$ and continuous on $C \cup C^+$.
2. $\sigma_2(z^{-1})$ is regular on $C^- = \{z \colon |z| > 1\}$ and continuous on $C \cup C^-$.
3. $\lim_{z \to \infty} \sigma_2(z^{-1}) = Q_2(0,0)$.

Equation (2.2) with the condition (2.3) provides us an easy Riemann-Hilbert boundary value problem (Dirichlet problem) (see [9] sections 14.4 and 14.5). We can solve it using Cauchy integral.
The target functions are written in integral forms as follows.

Proposition 2.4 (solution).

$$\sigma_1(z) = \frac{1}{2\pi i} \int_{\zeta \in C} \frac{G(\lambda \; Re(\zeta))}{\zeta - z} d\zeta + Q_2(0,0) \qquad \text{if } |z| < 1$$

and

$$\sigma_2(z) = \frac{1}{2\pi i} \int_{\zeta \in C} \frac{G(\lambda \; Re(\zeta))}{\zeta - z} d\zeta + Q_2(0,0) \qquad \text{if } |z| > 1$$

Thus, we have obtained $Q_1(z,0)$ and $Q_2(0,z)$.

3. Boundary value problem for original kernel

We now return to kernel of (1.9) and solve the boundary-value problem referring to the procedure in section 2.
The zero of kernel is the variety $V = \{(z_1, z_2) \colon z_1 z_2 - B_1^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2) B_2^*(\lambda - \lambda_1 z_1 - \lambda_2 z_2) = 0\}$.
Let $w_i \stackrel{\text{def}}{=} 2 r_i z_i$ for $i = 1,2$. Then (2.1) is translated into

$$\sigma_1\left(\frac{w_1}{2r_1}\right) - \sigma_2\left(\frac{w_2}{2r_2}\right) = G\left(\frac{1}{2}\lambda(w_1 + w_2)\right) \tag{3.1}$$

on V, provided that $|w_i| \leq 2r_i$ for $i = 1,2$.

We shall consider the intersection F of V with the plane L: $\{(w_1, w_2):\ w_1 = \overline{w}_2\}$.
Then F is a curve written as

$$F = \{w:\ |w|^2 = 4r_1 r_2 B_1^* B_2^*(\lambda(1 - Re(w)))\} \quad = \{w:\ w = e^{i\phi} 2\sqrt{r_1 r_2}\sqrt{B_1^* B_2^*(\lambda(1 - Re(w)))}\,\}. \tag{3.2}$$

Now we have the following:

<u>Lemma 3.3.</u>

1. *F is a smooth contour (diffeomorphic to a circle) around the origin.*

2. $\sigma_1(\frac{w}{2r_1})$ *is regular on F^+,the interior of F, and continuous on $F \cup F^+$.*

3. $\sigma_2(\frac{\overline{w}}{2r_2})$ *converges on $F \cup F_+$.*

4. $\sigma_1(\frac{w}{2r_1}) - \sigma_2(\frac{\overline{w}}{2r_2}) = G(\lambda Re(w))$ *on F.*

5. $\sigma_2(0) = Q_2(0,0)$.

Lemma 3.3 provides the conditions for Riemann-Hilbert boundary value problem with respect to contour F. In order to clarify the resemblance to our treatment in section 2 , we introduce conformal mapping

$f: z \in C^+ \ \mapsto\ w = f(z) \in F^+$

and its inverse

$f^{-1}: w \in F^+ \ \mapsto\ z = f^{-1}(w) \in C^+$

such that $f(0) = 0$.

Both f and f^{-1} can be canonically given by Theodorson's procedure (see [9] section 16.8), in which we must solve an integral equation.

Assuming that we have obtained the conformal mapping f, we proceed to define functions corresponding to $\sigma_1(z)$ and $\sigma_2(z)$ in observation 2.5. This can be done by choosing

$$\Sigma_1(z) \overset{\text{def}}{=} \sigma_1(\frac{f(z)}{2r_1}) \tag{3.4.1}$$

and

$$\Sigma_2(z) \overset{\text{def}}{=} \sigma_2(\frac{f(z^{-1})}{2r_2}). \tag{3.4.2}$$

Then we obtain

Observation 3.5.

1. $\Sigma_1(z)$ is regular with respect to z on the unit disk C^+ and continuous on $C \cup C^+$.
2. $\Sigma_2(z^{-1})$ is regular on C^- and continuous on $C \cup C^-$.
3. $\lim_{z \to \infty} \Sigma_2(z^{-1}) = Q_2(0,0)$.
4. $\Sigma_1(z) - \Sigma_2(z) = G(\lambda\, Re(f(z)))$ on the unit circle C.

This condition is identical to the combination of (2.2) and (2.3). Thus, we can get integral expressions of both Σ_1 and Σ_2 similar to (2.4).

Proposition 3.6.

$$\Sigma_1(z) = \frac{1}{2\pi i}\int_{\zeta \in C} \frac{G(\lambda\, Re(f(\zeta)))}{\zeta - z} d\zeta + Q_2(0,0) \qquad \text{if } |z| < 1$$

and

$$\Sigma_2(z) = \frac{1}{2\pi i}\int_{\zeta \in C} \frac{G(\lambda\, Re(f(\zeta)))}{\zeta - z} d\zeta + Q_2(0,0) \qquad \text{if } |z| > 1$$

Thus, we obtain $\sigma_1(z)$ and $\sigma_2(z)$, and then $Q_1(z,0)$ and $Q_2(0,z)$.

4. Evaluation of waiting time.

Finally, let us analyze the waiting time of customers. Because of equations (1.5) and (1.6), it suffices to evaluate $\frac{dQ_1(z,0)}{dz}\big|_{z=1}$ and $\frac{dQ_2(0,z)}{dz}\big|_{z=1}$ Without loss of generality, we assume that $r_1 \le r_2$.

From (1.8.1) and (3.4.1),

$$Q_1(z,0) = \frac{1}{2}\{\sigma_1(z) + Q_1(0,0)\} = \frac{1}{2}\{\Sigma_1(f^{-1}(2r_1 z)) + Q_1(0,0)\} \tag{4.1}$$

Since the domain of f^{-1} is F^+, we can evaluate $\frac{dQ_1(z,0)}{dz}\big|_{z=1}$ by using the integral form shown in proposition 3.6 if the following condition holds;

Condition 4.2. $2r_1$ is involved in F^+.

Therefore, provided with (4.2), we can evaluate $E(W_1)$.

The condition (4.2) holds for most of systems; for example , if the service times of queues are constants (b_1 and b_2), then $\lambda(b_1 + b_2) \le 2$ is a sufficient condition for (4.2).

Remark : If $2r_1$ stays out of F^+, we cannot get the differential $\frac{dQ_1(z,0)}{dz}\big|_{z=1}$ from proposition 3.6 directly. In sucha case, we evaluate it numerically from the Taylor expansion of $\Sigma_1(z)$, whose coefficients can be calculated from the integral formula given in proposition 3.6.

On the other hand, the analogy of (4.2) fails for any r_2, in other words, $2r_2$ always stays out of F. Therefore, we cannot evaluate $E(W_2)$ without calculating the Taylor expansion of $\Sigma_2(z)$ in general.

However, if $S_1 = S_2$, we have the following "pseudo conservation law" (see [11]):

$$\frac{\lambda_1 b_1}{\rho}E(W_1) + \frac{\lambda_2 b_2}{\rho}E(W_2) = \frac{\lambda_1 b_1^{(2)} + \lambda_2 b_2^{(2)}}{2(1-\rho)} + \frac{2s + \lambda s^{(2)}}{2(1+\lambda s)}. \tag{4.3}$$

Hence, we can evaluate $E(W_2)$ from $E(W_1)$ by using (4.3).

Remark. Identity (4.3) does not hold in general if $S_1 \neq S_2$.

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