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Complexity of the Police Officer Patrol Problem

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Abstract: We introduce an edge routing decision problem called the police officer patrol problem (POPP), which is related to the vertex cover problem. A vertex cover of a graph can be regarded as the placement of police officers or fixed surveillance cameras so that each street of a neighborhood represented by the graph can be confirmed visually without moving from their position. In the edge routing problem we consider, a single police officer must confirm all the streets. The officer is allowed to move, but can confirm any street visually from an incident intersection without traversing it. In this paper, we show that the POPP on mixed graphs is **NP**-complete.

Keywords: edge routing problem, vertex cover problem, mixed graph, **NP**-complete

1. Introduction

The Chinese postman problem (CPP) is to decide whether there exists a tour for a post officer in a given area within a given amount of time which starts and ends at the post office. The post officer must traverse every street in the area at least once. CPP is one of the typical edge routing problems on graphs introduced by Mei-Ko [7].

Edmonds and Johnson [1] showed that CPP on undirected graphs or directed graphs can be solved in polynomial time. On the other hand, Papadimitriou [8] showed that CPP on mixed graphs is **NP**-complete. Mixed graphs represent the realistic situation in urban areas where there are both two-way streets and one-way streets. He showed that CPP remains **NP**-complete even if restricted to those whose edges all have equal length or those on mixed planner graphs or on mixed graphs with vertices of degree three. Tohyama and Adachi [9] investigated how the complexity of CPP on a mixed graph changes with the addition of a limit on the number of times each edge can be traversed. Specifically, they showed that even if the number of traversals of each edge is restricted to two, CPP on mixed graphs remains **NP**-complete.

The rural postman problem (RPP) is one of the generalizations of CPP with a given set of edges that must be traversed by the postman. This problem focuses on the fact that in rural areas not every street has a delivery destination. Lenstra and Rinnoy-Kan [5], [6] showed that the optimization version of RPP on undirected graphs or directed graphs is **NP**-hard.

If the number of traversals of each edge is restricted to exactly one, CPP is equivalent to the Eulerian circuit problem. The Eulerian circuit problem on undirected graphs can be solved in polynomial time since it is only necessary to determine whether or not the degree of each vertex is even. Similarly, the Eulerian circuit problem on directed graphs can also be solved in polynomial time since it is only necessary to determine whether or not the indegree and outdegree of each vertex are equal.

In Ref. [9], it is also shown that the Eulerian circuit problem on mixed graphs can be solved in polynomial time. Let $G = (V, E, A)$ be a mixed graph. Here V is the set of vertices, E is the set of undirected edges and A is the set of directed edges. If there exists a vertex $v \in V$ such that $\deg(v) - |\mathrm{indeg}(v) - \mathrm{outdeg}(v)| \not\equiv \pmod 2$, then it is obvious that G does not have an Eulerian circuit. Otherwise, we construct the following bipartite graph $G' = (V_1, V_2, E')$:

- The set V_1 consists of the edges of G. That is, $V_1 = E \cup A$;
- The set V_2 consists of $\deg(v)/2$-copies of v for each vertex v of G;
- If $e \in E$ is incident to $v \in V$, then the vertex $e \in V_1$ is adjacent to each copy of v in V_2;
- If $a \in A$ is oriented to $v \in V$, then the vertex $a \in V_1$ is adjacent to each copy of v in V_2.

For instance, given the mixed graph illustrated in **Fig. 1** (i), we construct the bipartite graph illustrated in Fig. 1 (ii). By finding a perfect matching for this graph, an orientation of the undirected edges in G can be determined, giving an Eulerian circuit. The Eulerian circuit problem on mixed graphs can therefore be solved in polynomial time since this bipartite graph can be constructed in polynomial time and there exists an $O(n^{5/2})$ time algorithm for the perfect matching problem [3].

The vertex cover problem (VC) is a well known classical graph problem. A vertex cover of a graph is a set of vertices that includes at least one endpoint of every edge of the graph. VC is the problem of deciding whether there exists a vertex cover of size at most k in a given graph G where k is a given positive integer. VC is one of Karp's 21 **NP**-complete problems [4]. The connected vertex cover problem (CVC) is to decide whether there exists a vertex cover V' of size at most k such that the subgraph induced by V' is connected for a given graph and positive integer k. CVC is **NP**-complete problem introduced by Garey and Johnson [2] and they also showed that CVC on planar graphs of maximum degree 4 remains **NP**-complete.

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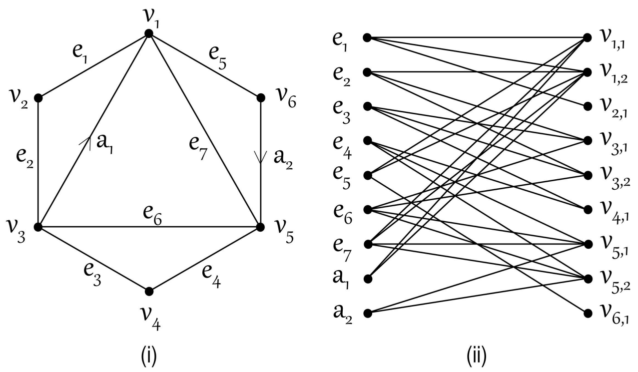

Fig. 1 (i) Mixed graph and (ii) the corresponding bipartite graph.

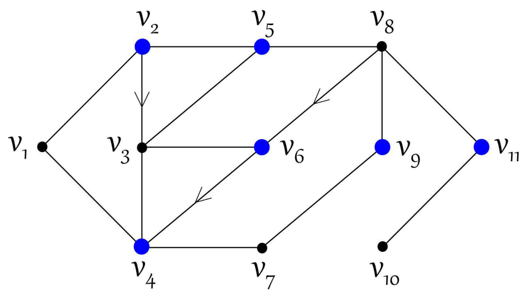

Fig. 2 Mixed graph. The set of blue vertices is one of vertex covers of this graph.

In urban settings, crime prevention measures taking the whole area into consideration are especially important. From this point of view, a vertex cover of a graph can be regarded as an appropriate placement of police officers or fixed surveillance cameras. A police officer placed at an intersection (vertex) can confirm each incident street (edges) visually without moving. Not only two-way streets but also one-way streets can also be confirmed confirmed visually from either of its incident intersections. We therefore consider a vertex cover of a mixed graph to be a vertex cover of the underlying undirected graph obtained by ignoring the orientation of its directed edges. For instance, given the mixed graph illustrated in **Fig. 2**, the set $\{v_2, v_4, v_5, v_6, v_9, v_{11}\}$ of vertices is a vertex cover.

In this paper, we introduce an edge routing decision problem which is to find a patrol route for one police officer to confirm all streets. The police officer is allowed to confirm any street visually from an incident intersection without traversing it. Therefore, he does not have to traverse all the streets. That is, the set of vertices on the patrol route is a vertex cover for the given graph. The police officer patrol problem (POPP) is the problem of deciding whether there exists a patrol route for a given mixed graph in which each edge is either traversed exactly once or confirmed visually. In this paper, we show that POPP is **NP**-complete by a similar method to the one used in Ref. [9].

2. Police Officer Patrol Problem (POPP)

Let $G = (V, E, A)$ be a connected simple mixed graph. Throughout this paper, when we simply refer to an "edge," we mean either an undirected or a directed edge. A sequence $\mathfrak{S} : v_0, v_1, v_2, \cdots, v_n$ of vertices is said to be a patrol route on G, if the following conditions hold:

(1) For each i $(0 \le i < n)$, either $\{v_i, v_{i+1}\} \in E$ or $(v_i, v_{i+1}) \in A$.

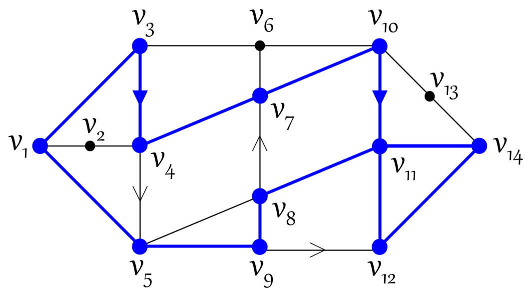

Fig. 3 Mixed graph and a patrol route $\mathfrak{S} : v_1, v_3, v_4, v_7, v_{10}, v_{11}, v_{14}, v_{12}, v_{11}, v_8, v_9, v_5$ and v_1. The set of vertices $R_{\mathfrak{S}} = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{14}\}$ in $\mathfrak{S}$ is one of vertex covers of this graph.

(2) For any i and j $(0 \le i < j < n)$,

$$\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\} \ \ \textit{if} \ \ \{v_i, v_{i+1}\}, \{v_j, v_{j+1}\} \in E$$

and

$$(v_i, v_{i+1}) \neq (v_j, v_{j+1}) \ \ \textit{if} \ \ (v_i, v_{i+1}), (v_j, v_{j+1}) \in A.$$

(3) The set $R_{\mathfrak{S}} = \{v_i : 0 \le i \le n\}$ of vertices in $\mathfrak{S}$ is a vertex cover of G. That is, $R_{\mathfrak{S}} \cap \{v, v'\} \neq \phi$ holds for any $(v, v') \in A$ and $\{v, v'\} \in E$.

(4) $v_0 = v_n$.

Here, (1) means that there exists an edge between two successive vertices in $\mathfrak{S}$. Directed edges must be traversed according to their direction. It is said that each edge $\{v_i, v_{i+1}\} \in E$ (or $(v_i, v_{i+1}) \in A$) between members of $\mathfrak{S}$ is traversed from v_i to v_{i+1}. (2) means that the same edge cannot be traversed more than once. (3) means that for any edge e, at least one vertex to which e is incident is in $\mathfrak{S}$. If exactly one vertex to which e is incident is in $\mathfrak{S}$, it is said that e is confirmed visually. By (1) and (4), $\mathfrak{S}$ is a circuit.

For instance, we illustrate a mixed graph and a patrol route on it in **Fig. 3**.

The POPP is to decide whether there exists a patrol route for a given mixed graph.

3. Result

In order to show that POPP is **NP**-complete, we introduce four graphs illustrated on the left in **Fig. 4**. These will appear as subgraphs in a graph constructed in our proof of **NP**-completeness. In each of these graphs, only v and v' may have additional neighbors outside of each graph. Therefore, we call v and v' external vertices. We call u_i internal vertices since they have no neighbors outside of each graph.

For each graph in Fig. 4, consider a patrol route for a graph containing it as a subgraph:

(1) In the subgraph (i), both directed edges (v, u_2) and (u_2, v') must be traversed in order to confirm the undirected edge $\{u_1, u_2\}$ visually. Therefore, we regard this subgraph as a deemed edge with end endpoints v and v' which must be traversed from v to v' exactly once. We call the subgraph (i) an α-edge and denote it by $(v, v')^{\rightarrow}$.

(2) In the subgraph (ii), both undirected edges $\{v, u_2\}$ and $\{u_2, v'\}$ must be traversed in order to confirm the undirected edge $\{u_1, u_2\}$ visually. We can choose whether these edges are traversed either from v to u_2 and from u_2 to v' respectively or from u_2 to v and from v' to u_2 respectively. Therefore, we regard this subgraph as a deemed edge with endpoints v and

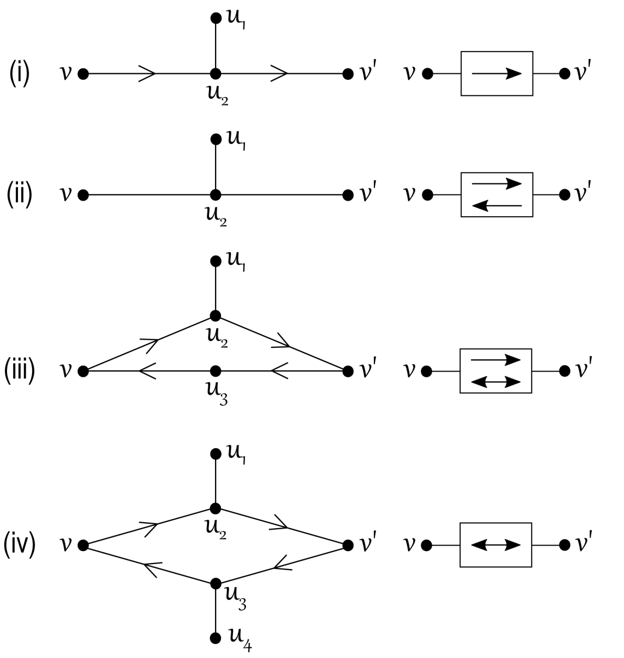

Fig. 4 (i) α-edge $(v, v')^{\rightarrow}$, (ii) β-edge $(v, v')^{\rightarrow}_{\leftarrow}$, (iii) γ-edge $(v, v')^{\rightarrow}_{\leftrightarrow}$ and (iv) δ-edge $(v, v')^{\leftrightarrow}$, and their representations.

v' which must be traversed either from v to v' or from v' to v exactly once. We call the subgraph (ii) a β-edge and denote it by $(v, v')^{\rightarrow}_{\leftarrow}$.

(3) In the graph (iii), both directed edges (v, u_2) and (u_2, v') must be traversed in order to confirm the undirected edge $\{u_1, u_2\}$ visually. On the other hand, we can choose whether (v', u_3) and (u_3, v) are confirmed visually or traversed. When we regard this subgraph as a deemed edge, we say that the deemed edge is traversed as a round trip between v and v', if the patrol route includes u_3. That is, we regard this subgraph as a deemed edge with endpoints v and v' which is either traversed from v to v' exactly once or traversed as a round trip between v and v' exactly once. We call the subgraph (iii) a γ-edge and denote it by $(v, v')^{\rightarrow}_{\leftrightarrow}$.

(4) In the subgraph (iv), both directed edges (v, u_2) and (u_2, v') must be traversed in order to confirm the undirected edge $\{u_1, u_2\}$ visually, and (v', u_3) and (u_3, v) must also be traversed in order to confirm the undirected edge $\{u_3, u_4\}$ visually. Therefore, we regard this subgraph as a deemed edge with endpoints v and v' which is traversed as a round trip between v and v' exactly once. We call the subgraph (iv) a δ-edge and denote it by $(v, v')^{\leftrightarrow}$.

We represent an α-edge $(v, v')^{\rightarrow}$, a β-edge $(v, v')^{\rightarrow}_{\leftarrow}$, a γ-edge $(v, v')^{\rightarrow}_{\leftrightarrow}$ and a δ-edge $(v, v')^{\leftrightarrow}$ as shown on the right in Fig. 4. To simplify our discussion, we treat these deemed edges as normal edges below. That is, when we give the definition of a graph using deemed edges, only their external vertices are indicated, and not their internal vertices and edges. Additionally, when we define a mixed graph $G = (V, E, A)$ using deemed edges, we do not distinguish between the set of undirected edges E and the set of directed edges A, and simply write $G = (V, E)$.

First, we give a property of a patrol route in a graph with three deemed edges incident to the same vertex.

Lemma 1 Let G be any mixed graph including one β-edge $(v, v_1)^{\rightarrow}_{\leftarrow}$ and two γ-edges $(v_2, v)^{\rightarrow}_{\leftrightarrow}$, $(v, v_3)^{\rightarrow}_{\leftrightarrow}$ which are incident to a vertex v (**Fig. 5**). Suppose that only these three deemed edges are incident to v. Then, for any patrol route $\mathfrak{S}$ on G, either of the following holds:

(1) $(v, v_1)^{\rightarrow}_{\leftarrow}$ is traversed from v to v_1, $(v_2, v)^{\rightarrow}_{\leftrightarrow}$ is traversed from v_2 to v and $(v, v_3)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip.

(2) $(v, v_1)^{\rightarrow}_{\leftarrow}$ is traversed from v_1 to v, $(v_2, v)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip and $(v, v_3)^{\rightarrow}_{\leftrightarrow}$ is traversed from v to v_3.

This assertion holds even if two γ-edges are incident to the same two vertices (that is, $v_2 = v_3$).

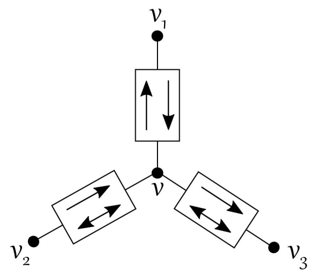

Fig. 5 Three deemed edges $(v, v_1)^{\rightarrow}_{\leftarrow}$, $(v_2, v)^{\rightarrow}_{\leftrightarrow}$ and $(v, v_3)^{\rightarrow}_{\leftrightarrow}$ which are incident to the same vertex v.

Proof This assertion is obtained by considering the number of in-going and out-going traversals at v in $\mathfrak{S}$. □

In our proof of **NP**-completeness for POPP, we show that 3SAT (Ref. [4]) is reducible to POPP in polynomial time. We construct graphs corresponding to the variables and clauses which arise in a given Boolean formula in 3-conjunctive normal form, and show their properties in advance.

For a non-negative integer z and a label σ, we define a set of vertices V^z_σ and a set of edges E^z_σ as follows:

$$V^z_\sigma = \begin{cases} \{\sigma_0, \sigma'_0\}, & \text{if } z = 0, \\ \{\sigma_i, \sigma'_i : 1 \le i \le z\}, & \text{otherwise,} \end{cases}$$

$$E^z_\sigma = \begin{cases} \{(\sigma_0, \sigma'_0)^{\rightarrow}_{\leftrightarrow}, (\sigma'_0, \sigma_0)^{\rightarrow}_{\leftrightarrow}\}, & \text{if } z = 0, \\ \{(\sigma_i, \sigma'_i)^{\rightarrow}_{\leftrightarrow} : 1 \le i \le z\} \\ \cup \{(\sigma'_i, \sigma_{i+1})^{\rightarrow}_{\leftarrow} : 1 \le i < z\}, & \text{otherwise.} \end{cases}$$

These are used to construct a mixed graph corresponding to each variable. Let s and t be non-negative integers with $s + t \ge 1$ and let x be a label. Then we define a mixed graph $G^{s,t}_x = (V^{s,t}_x, E^{s,t}_x)$. Here, $V^{s,t}_x = V^s_x \cup V^t_{\overline{x}}$ and $E^{s,t}_x = E^s_x \cup E^t_{\overline{x}} \cup \hat{E}^{s,t}_x$, where

$$\hat{E}^{s,t}_x = \begin{cases} \{(x_0, \overline{x}_1)^{\rightarrow}_{\leftarrow}, (x'_0, \overline{x}'_t)^{\rightarrow}_{\leftarrow}\}, & \text{if } s = 0, \\ \{(x_1, \overline{x}_0)^{\rightarrow}_{\leftarrow}, (x'_s, \overline{x}'_0)^{\rightarrow}_{\leftarrow}\}, & \text{if } t = 0, \\ \{(x_1, \overline{x}_1)^{\rightarrow}_{\leftarrow}, (x'_s, \overline{x}'_t)^{\rightarrow}_{\leftarrow}\}, & \text{otherwise.} \end{cases}$$

Furthermore, in order to show a property of $G^{s,t}_x$, we define a mixed graph $\tilde{G}^{s,t}_x = (\tilde{V}^{s,t}_x, \tilde{E}^{s,t}_x)$ which contains $G^{s,t}_x$ as a subgraph. Here, let y be an arbitrary fixed label,

$$\tilde{V}^{s,t}_x = \begin{cases} V^{0,t}_x \cup V^t_{\overline{y}}, & \text{if } s = 0, \\ V^{s,0}_x \cup V^s_y, & \text{if } t = 0, \\ V^{s,t}_x \cup V^{s,t}_y, & \text{otherwise,} \end{cases}$$

and

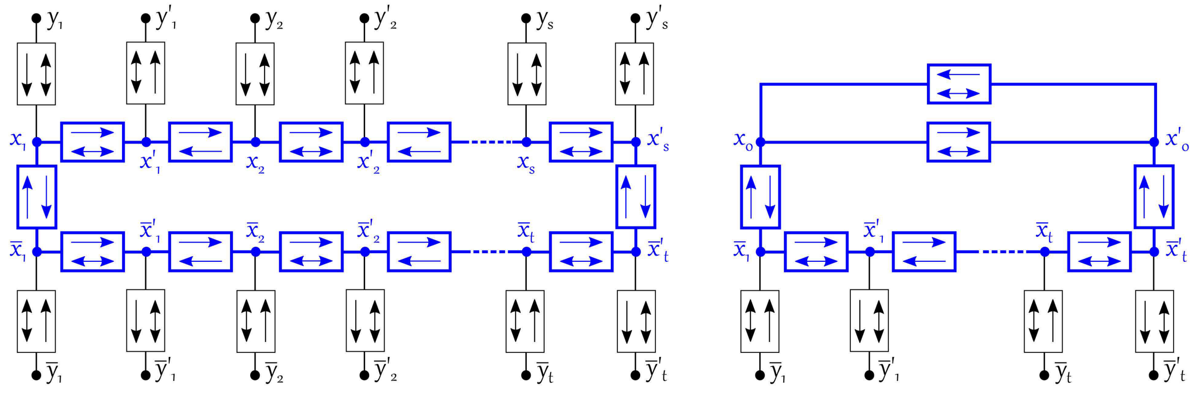

Fig. 6 Mixed graphs (i) $\tilde{G}^{s,t}_x$ $(s,t \geq 1)$ and (ii) $\tilde{G}^{0,t}_x$. Blue parts are $G^{s,t}_x$ and $G^{0,t}_x$.

$$\tilde{E}^{s,t}_x = E^{s,t}_x \cup \{(y_i,x_i)^{\rightarrow}_{\leftrightarrow}, (x'_i,y'_i)^{\rightarrow}_{\leftrightarrow} : 1 \leq i \leq s\}$$
$$\cup \{(\overline{y}_i,\overline{x}_i)^{\rightarrow}_{\leftrightarrow}, (\overline{x}'_i,\overline{y}'_i)^{\rightarrow}_{\leftrightarrow} : 1 \leq i \leq t\}.$$

We illustrate the graph $\tilde{G}^{s,t}_x$ and its subgraph $G^{s,t}_x$ in **Fig. 6**.

For the mixed graph $\tilde{G}^{s,t}_x$, the following lemma holds.

Lemma 2 Let G be a mixed graph with a subgraph isomorphic to $\tilde{G}^{s,t}_x$. Assume that only the vertices y_i, y'_i, $\overline{y}_i$ and $\overline{y}'_i$ have neighbors outside $\tilde{V}^{s,t}_x$ (that is, they are external vertices of $\tilde{G}^{s,t}_x$). Then for any patrol route $\mathfrak{S}$ on G, either of the following conditions is satisfied:

(1) Every γ-edge $(x_i,x'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed from x_i to x'_i and every γ-edge $(\overline{x}_i,\overline{x}'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip.

(2) Every γ-edge $(x_i,x'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip and every γ-edge $(\overline{x}_i,\overline{x}'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed from $\overline{x}_i$ to $\overline{x}'_i$.

Proof First, since every vertex in $V^{s,t}_x$ is incident to one β-edge and two γ-edges in directions opposite each other, we remark that every edge-induced subgraph generated by the set of these three deemed edges is isomorphic to the graph constructed in Lemma 1. If $s=0$ or $t=0$, subgraphs that are isomorphic to the one in Fig. 5 with $v_2 = v_3$ are included in the collection of these edge-induced subgraphs (For instance, in the case $s=0$ (Fig. 6 (ii)), two edge-induced subgraphs generated by the set of $(x_0,\overline{x}_1)^{\rightarrow}_{\leftarrow}$, $(x_0,x'_0)^{\rightarrow}_{\leftrightarrow}$ and $(x'_0,x_0)^{\rightarrow}_{\leftrightarrow}$ and the set of $(x'_0,\overline{x}'_t)^{\rightarrow}_{\leftarrow}$, $(x_0,x'_0)^{\rightarrow}_{\leftrightarrow}$ and $(x'_0,x_0)^{\rightarrow}_{\leftrightarrow}$ are included).

We focus on the deemed edge $(x'_s,\overline{x}'_t)^{\rightarrow}_{\leftarrow}$. Suppose that it is traversed from x'_s to $\overline{x}'_t$ by $\mathfrak{S}$. Since the way to traverse this edge is decided, by Lemma 1, the way to traverse the remaining two deemed edges incident to x'_s is also decided. That is, $(x_s,x'_s)^{\rightarrow}_{\leftrightarrow}$ must be traversed from x_s to x'_s and $(x'_s,y'_s)^{\rightarrow}_{\leftrightarrow}$ must be traversed as a round trip ($(x'_0,x_0)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip if $s=0$). Similarly, how to traverse the remaining two deemed edges which are incident to $\overline{x}'_t$ can also be decided. That is, $(\overline{x}_t,\overline{x}'_t)^{\rightarrow}_{\leftrightarrow}$ must be traversed as a round trip and $(\overline{x}'_t,\overline{y}'_t)^{\rightarrow}_{\leftrightarrow}$ must be traversed from $\overline{x}'_t$ to $\overline{y}'_t$ ($(\overline{x}'_0,\overline{x}_0)^{\rightarrow}_{\leftrightarrow}$ is traversed from $\overline{x}'_0$ to $\overline{x}_0$ if $t=0$). Moreover, the decision on how to traverse $(x_s,x'_s)^{\rightarrow}_{\leftrightarrow}$ and $(\overline{x}_t,\overline{x}'_t)^{\rightarrow}_{\leftrightarrow}$, determines how to traverse the remaining deemed edges which are incident to x_s and $\overline{x}_t$ is decided. By applying Lemma 1 repeatedly in this way, if $(x'_s,\overline{x}'_t)^{\rightarrow}_{\leftarrow}$ is traversed from x'_s to $\overline{x}'_t$, then we obtain the fact that every γ-edge $(x_i,x'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed from x_i to x'_i and every γ-edge $(\overline{x}_i,\overline{x}'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip. By a similar argument, we can verify that every γ-edge $(x_i,x'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip and every γ-edge $(\overline{x}_i,\overline{x}'_i)^{\rightarrow}_{\leftrightarrow}$ is traversed from $\overline{x}_i$ to $\overline{x}'_i$ if $(x'_s,\overline{x}'_t)^{\rightarrow}_{\leftarrow}$ is traversed from $\overline{x}'_t$ to x'_s. □

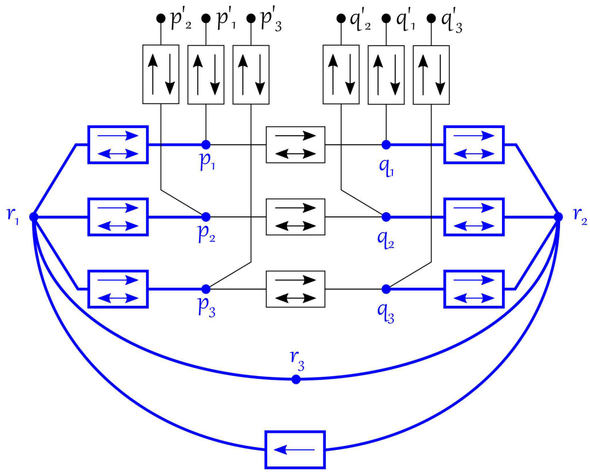

Fig. 7 Mixed graph $\tilde{G}_C$ and its subgraph G_C (blue part).

Next, we construct a mixed graph corresponding to a clause in a Boolean formula in 3-conjunctive normal form. Let C be a label and p, q and r be fixed labels determined by C. We define a mixed graph $G_C = (V_C, E_C)$, where $V_C = \{p_j, q_j, r_j : 1 \leq j \leq 3\}$ and

$$E_C = \{(r_1,p_j)^{\rightarrow}_{\leftrightarrow}, (q_j,r_2)^{\rightarrow}_{\leftrightarrow} : 1 \leq j \leq 3\}$$
$$\cup \{\{r_1,r_3\},\{r_2,r_3\},(r_2,r_1)^{\rightarrow}\}.$$

Furthermore, in order to describe a property of G_C, we define a mixed graph $\tilde{G}_C = (\tilde{V}_C, \tilde{E}_C)$ which contains G_C as a subgraph. Here,

$$\tilde{V}_C = V_C \cup \{p'_j, q'_j : 1 \leq j \leq 3\}$$
$$\tilde{E}_C = E_C \cup \{(p_j,q_j)^{\rightarrow}_{\leftrightarrow}, (p_j,p'_j)^{\rightarrow}_{\leftarrow}, (q_j,q'_j)^{\rightarrow}_{\leftarrow} : 1 \leq j \leq 3\}.$$

We illustrate $\tilde{G}_C$ and its subgraph G_C in **Fig. 7**.

For the mixed graph $\tilde{G}_C$, the following lemma holds.

Lemma 3 Let G be a mixed graph with $\tilde{G}_C$ as a subgraph. Suppose that only the vertices p'_j and q'_j have neighbors outside $\tilde{V}_C$ (In other words, these are external vertices of $\tilde{G}_C$). Then for any patrol route $\mathfrak{S}$ on G, at least one of the three γ-edges $(p_j,q_j)^{\rightarrow}_{\leftrightarrow}$

must be traversed from p_j to q_j.

Proof The vertices p_j and q_j are each incident to one β-edge and two γ-edges in opposite directions. Therefore, by Lemma 1, if the way to traverse one of these three deemed edges is decided, the way to traverse the others is also decided, accordingly. Specifically, if $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ is traversed from p_j to q_j, then:

- □ $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed as a round trip and $(p_j, p'_j)^{\rightleftarrows}_{\leftarrow}$ must be traversed from p'_j to p_j;
- □ $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed as a round trip and $(q_j, q'_j)^{\rightleftarrows}_{\leftarrow}$ must be traversed from q_j to q'_j.

Similarly, if $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ is traversed as a round trip, then:

- □ $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed from r_1 to p_j and $(p_j, p'_j)^{\rightleftarrows}_{\leftarrow}$ must be traversed from p_j to p'_j;
- □ $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed from q_j to r_2 and $(q_j, q'_j)^{\rightleftarrows}_{\leftarrow}$ must be traversed from q'_j to q_j.

Suppose that k γ-edges $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ are traversed from p_j to q_j on $\mathfrak{S}$ (the remaining $3 - k$ γ-edges $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ are traversed as round trips). Then, by the relation between the number of in-going and out-going traversals at r_1 and r_2, the following statements hold:

(i) if $k = 3$, for each j $(1 \le j \le 3)$, both $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed as a round trip. In this case, it is necessary that $\{r_1, r_3\}$ is traversed from r_1 to r_3 and $\{r_2, r_3\}$ is traversed from r_3 to r_2.

(ii) if $k = 2$ then, for exactly one value of j $(1 \le j \le 3)$, $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed from r_1 to p_j and from q_j to r_2, respectively. For the two remaining values of j, both $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed as round trips. In this case, it is necessary that both $\{r_1, r_3\}$ and $\{r_3, r_2\}$ are confirmed visually.

(iii) if $k = 1$, for exactly two values of j $(1 \le j \le 3)$, $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed from r_1 to p_j and from q_j to r_2, respectively. For the remaining value of j, both $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed as round trips. In this case, it is necessary that $\{r_2, r_3\}$ is traversed from r_2 to r_3 and $\{r_1, r_3\}$ is traversed from r_3 to r_1.

(iv) If $k = 0$, that is, all three γ-edges $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ are traversed as round trips, then for each j $(1 \le j \le 3)$, $(r_1, p_j)^{\rightleftarrows}_{\leftrightarrow}$ and $(q_j, r_2)^{\rightleftarrows}_{\leftrightarrow}$ must be traversed from r_1 to p_j and from q_j to r_2, respectively. In this case, regardless of whether the edges $\{r_1, r_3\}$ and $\{r_2, r_3\}$ are traversed or confirmed visually, the number of in-going and out-going traversals at r_1 (at r_2 similarly) cannot be equal. Hence, there exists no patrol route on G such that all three γ-edges $(p_j, q_j)^{\rightleftarrows}_{\leftrightarrow}$ are traversed as round trips. □

Theorem POPP is **NP**-complete.

Proof It is obvious that POPP is in **NP**. It remains for us to show that 3SAT is reducible to POPP in polynomial time. That is, for any Boolean formula F in 3-conjunctive normal form, we show that a mixed graph G_F which satisfies the following condition can be constructed in polynomial time:

$$F \text{ is satisfiable} \iff G_F \text{ has a patrol route.}$$

Let $F = C_1 \cdot C_2 \cdot \cdots \cdot C_m$ be a Boolean formula in 3-conjunctive normal form with n variables $x_1, x_2, \cdots, x_n$, where

$$C_i = c_{i1} + c_{i2} + c_{i3} \quad (1 \le i \le m)$$

and each c_{ij} $(1 \le j \le 3)$ is either a variable or its negation. Let s_k and t_k be the number of appearances of x_k and $\overline{x}_k$ in F, respectively. For any k $(1 \le k \le n)$, $s_k + t_k \ge 1$ holds, since F contains at least one x_k or $\overline{x}_k$. Moreover, we assume that s_k literals $c_{i_{k,1} j_{k,1}}$, $c_{i_{k,2} j_{k,2}}$, $\cdots$, $c_{i_{k,s_k} j_{k,s_k}}$ are equal to x_k, and t_k literals $c_{i'_{k,1} j'_{k,1}}$, $c_{i'_{k,2} j'_{k,2}}$, $\cdots$, $c_{i'_{k,t_k} j'_{k,t_k}}$ are equal to $\overline{x}_k$. Suppose that their indices satisfy the following conditions:

- □ $i_{k,1} \le i_{k,2} \le \cdots \le i_{k,s_k}$,
- □ $j_{k,l} < j_{k,l+1}$ *if* $i_{k,l} = i_{k,l+1}$,
- □ $i'_{k,1} \le i'_{k,2} \le \cdots \le i'_{k,t_k}$,
- □ $j'_{k,l} < j'_{k,l+1}$ *if* $i'_{k,l} = i'_{k,l+1}$.

We construct a mixed graph $G_F = (V_F, E_F)$ from the formula F. Here, G_F contains the following two kinds of subgraphs:

- □ $G^{s_k,t_k}_{x_k} = (V^{s_k,t_k}_{x_k}, E^{s_k,t_k}_{x_k})$ corresponding to each variable x_k $(1 \le k \le n)$. $G^{s_k,t_k}_{x_k}$ is the subgraph of the mixed graph $\tilde{G}^{s_k,t_k}_{x_k}$ constructed in Lemma 2.
- □ $G_{C_i} = (V_{C_i}, E_{C_i})$ corresponding to each clause C_i $(1 \le i \le m)$. Here,

$$\begin{aligned} V_{C_i} &= \{p_{i,j}, q_{i,j}, r_{i,j} : 1 \le j \le 3\}, \\ E_{C_i} &= \{(r_{i,1}, p_{i,j})^{\rightleftarrows}_{\leftrightarrow}, (q_{i,j}, r_{i,2})^{\rightleftarrows}_{\leftrightarrow} : 1 \le j \le 3\} \\ &\quad \cup \{\{r_{i,1}, r_{i,3}\}, \{r_{i,2}, r_{i,3}\}, (r_{i,2}, r_{i,1})^{\rightarrow}\}. \end{aligned}$$

We remark that G_{C_i} is isomorphic to the graph G_C constructed in Lemma 3.

We combine n mixed graphs $G^{s_k,t_k}_{x_k}$ and m mixed graphs G_{C_i} by equating vertices of $G^{s_k,t_k}_{x_k}$ with vertices of G_{C_i} as follows: For each k $(1 \le k \le n)$,

- □ if $s_k > 0$, then for each l $(1 \le l \le s_k)$, the vertices $x_{k,l}$ and $x'_{k,l}$ in the subgraph $G^{s_k,t_k}_{x_k}$ are equated with the vertices $p_{i_{k,l},j_{k,l}}$ and $q_{i_{k,l},j_{k,l}}$ in the subgraph $G_{C_{i_{k,l}}}$, respectively.
- □ if $t_k > 0$, then for each l $(1 \le l \le t_k)$, the vertices $\overline{x}_{k,l}$ and $\overline{x}'_{k,l}$ in the subgraph $G^{s_k,t_k}_{x_k}$ are equated with the vertices $p_{i'_{k,l},j'_{k,l}}$ and $q_{i'_{k,l},j'_{k,l}}$ in the subgraph $G_{C_{i'_{k,l}}}$, respectively.

These equated vertices have two labels (for example, $x_{k,l}$ and $p_{i_{k,l},j_{k,l}}$), and we use these labels interchangeably according to convenience. Therefore, for instance, if vertices $x_{k,l}$ and $x'_{k,l}$ (or $\overline{x}_{k,l}$ and $\overline{x}'_{k,l}$) in $G^{s_k,t_k}_{x_k}$ are equated with vertices $p_{i_{k,l},j_{k,l}}$ and $q_{i_{k,l},j_{k,l}}$ in $G_{C_{i_{k,l}}}$, respectively, then the γ-edge $(x_{k,l}, x'_{k,l})^{\rightleftarrows}_{\leftrightarrow}$ (or $(\overline{x}_{k,l}, \overline{x}'_{k,l})^{\rightleftarrows}_{\leftrightarrow}$) is sometimes represented by $(p_{i_{k,l},j_{k,l}}, q_{i_{k,l},j_{k,l}})^{\rightleftarrows}_{\leftrightarrow}$.

The set of vertices V_F and the set of edges E_F of G_F are defined as follows:

$$\begin{aligned} V_F &= \left(\bigcup_{k=1}^{n} V^{s_k,t_k}_{x_k} \right) \cup \left(\bigcup_{i=1}^{m} V_{C_i} \right), \\ E_F &= \left(\bigcup_{k=1}^{n} E^{s_k,t_k}_{x_k} \right) \cup \left(\bigcup_{i=1}^{m} E_{C_i} \right) \cup \{(r_{i,1}, r_{i+1,1})^{\leftrightarrow} : 1 \le i < m\}. \end{aligned}$$

The vertices $r_{i,1}$ in G_{C_i} and $r_{i+1,1}$ in $G_{C_{i+1}}$ are connected by the δ-edge $(r_{i,1}, r_{i+1,1})^{\leftrightarrow}$. Since all δ-edges must be traversed as round trips, these have no effect on the relationship between the numbers of in-going and out-going traversals at their endpoints. Therefore, we can apply Lemma 3 for every subgraph in G_F which is isomorphic to the graph $\tilde{G}_C$ constructed in Lemma 3. The role of δ-edges is to guarantee that G_F is connected.

For instance, consider the Boolean formula $F = (\overline{x}_1 + x_2 + \overline{x}_3)(x_1 + \overline{x}_3 + x_4)(x_1 + x_3 + \overline{x}_4)$. We illustrate the mixed graphs

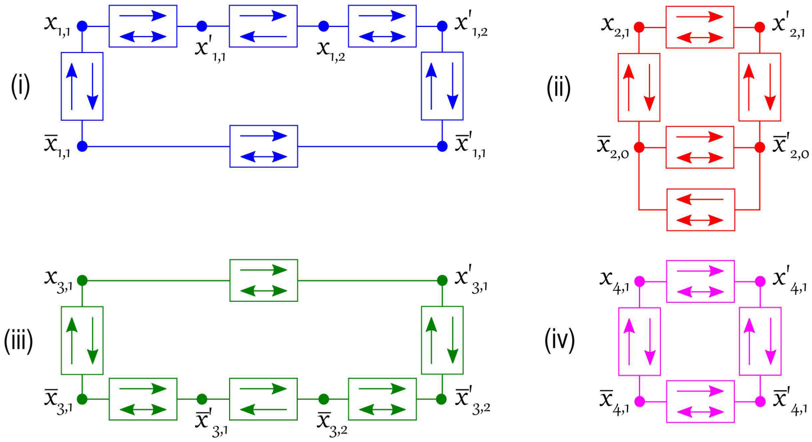

Fig. 8 Mixed graphs (i) $G^{2,1}_{x_1}$, (ii) $G^{1,0}_{x_2}$, (iii) $G^{1,2}_{x_3}$ and (iv) $G^{1,1}_{x_4}$.

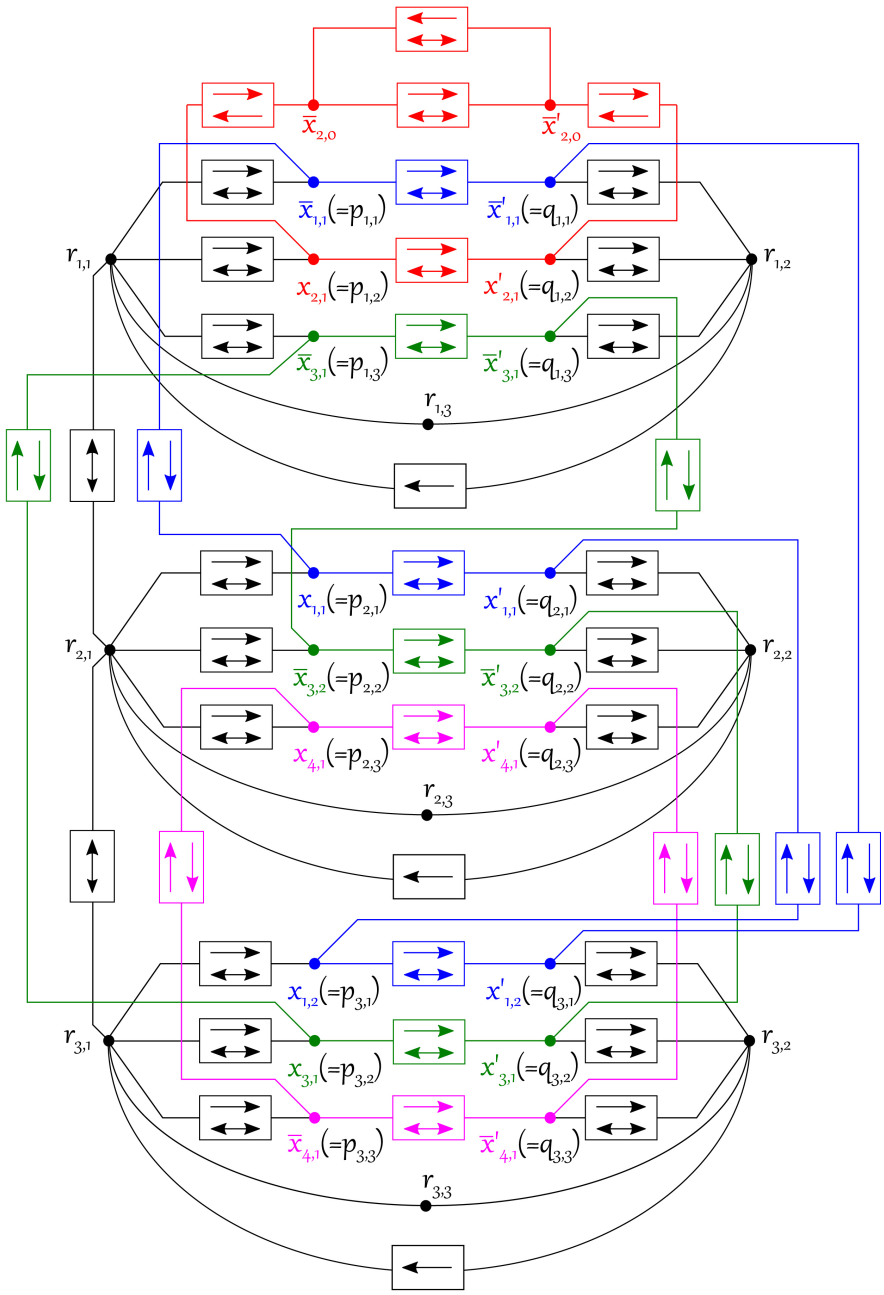

Fig. 9 Mixed graph G_F constructed from a Boolean formula in 3-conjunctive normal form $F = (\overline{x}_1 + x_2 + \overline{x}_3)(x_1 + \overline{x}_3 + x_4)(x_1 + x_3 + \overline{x}_4)$.

$G^{2,1}_{x_1}$, $G^{1,0}_{x_2}$, $G^{1,2}_{x_3}$ and $G^{1,1}_{x_4}$ corresponding to the variables x_1, x_2, x_3 and x_4 in **Fig. 8** and illustrate the graph G_F constructed from F in **Fig. 9**. Since the two literals c_{21} and c_{31} in F are equal to x_1 (that is, $i_{1,1} = 2$, $j_{1,1} = 1$, $i_{1,2} = 3$ and $j_{1,2} = 1$), the vertices $x_{1,1}$ and $x'_{1,1}$ in $G^{2,1}_{x_1}$ are equated with the vertices $p_{i_{1,1},j_{1,1}} (= p_{2,1})$ and $q_{i_{1,1},j_{1,1}} (= q_{2,1})$ in $G_{C_{i_{1,1}}} (= G_{C_2})$, respectively, and $x_{1,2}$ and $x'_{1,2}$ in $G^{2,1}_{x_1}$ are equated with $p_{i_{1,2},j_{1,2}} (= p_{3,1})$ and $q_{i_{1,2},j_{1,2}} (= q_{3,1})$ in $G_{C_{i_{1,2}}} (= G_{C_3})$, respectively. In addition, the remaining vertices of G_{x_k} are equated with the remaining vertices of G_{C_i} as follows:

$$\overline{x}_{1,1} = p_{1,1},\ \overline{x}'_{1,1} = q_{1,1},$$
$$x_{2,1} = p_{1,2},\ x'_{2,1} = q_{1,2},$$
$$x_{3,1} = p_{3,2},\ x'_{3,1} = q_{3,2},$$
$$\overline{x}_{3,1} = p_{1,3},\ \overline{x}'_{3,1} = q_{1,3},\quad \overline{x}_{3,2} = p_{2,2},\ \overline{x}'_{3,2} = q_{2,2},$$
$$x_{4,1} = p_{2,3},\ x'_{4,1} = q_{2,3},$$
$$\overline{x}_{4,1} = p_{3,3},\ \overline{x}'_{4,1} = q_{3,3}.$$

We remark that G_F includes subgraphs which are isomorphic to $\tilde{G}^{s_k,t_k}_{x_k}$ and $\tilde{G}_{C_i}$ for each variable x_k and clause C_i, respectively. For instance, in the graph G_F constructed from the formula $F = (\overline{x}_1 + x_2 + \overline{x}_3)(x_1 + \overline{x}_3 + x_4)(x_1 + x_3 + \overline{x}_4)$ (Fig. 8), the graph which adds the following six γ-edges to $G^{2,1}_{x_1}$ is isomorphic to $\tilde{G}^{2,1}_{x_1}$:

$$(r_{2,1}, x_{1,1})^{\rightarrow}_{\leftrightarrow},\ (x'_{1,1}, r_{2,2})^{\rightarrow}_{\leftrightarrow},\ (r_{3,1}, x_{1,2})^{\rightarrow}_{\leftrightarrow},$$
$$(x'_{1,2}, r_{3,2})^{\rightarrow}_{\leftrightarrow},\ (r_{1,1}, \overline{x}_{1,1})^{\rightarrow}_{\leftrightarrow},\ (\overline{x}'_{1,1}, r_{1,2})^{\rightarrow}_{\leftrightarrow}.$$

On the other hand, the graph which adds the following six β-edges to G_{C_1} is isomorphic to $\tilde{G}_{C_1}$:

$$(\overline{x}_{1,1}, x_{1,1})^{\rightarrow}_{\leftarrow},\ (\overline{x}'_{1,1}, \overline{x}'_{1,2})^{\rightarrow}_{\leftarrow},\ (x_{2,1}, \overline{x}_{2,0})^{\rightarrow}_{\leftarrow},$$
$$(x'_{2,1}, \overline{x}'_{2,0})^{\rightarrow}_{\leftarrow},\ (\overline{x}_{3,1}, x_{3,1})^{\rightarrow}_{\leftarrow},\ (\overline{x}'_{3,1}, x'_{1,1})^{\rightarrow}_{\leftarrow}.$$

It is obvious that G_F can be constructed from F in polynomial time. It remains for us to show that F is satisfiable if and only if G_F has a patrol route.

Let $I : \{x_1, x_2, \cdots, x_n\} \rightarrow \{0, 1\}$ be a truth assignment of F. Then, the β-edge $(x'_{k,s}, \overline{x}'_{k,t})^{\rightarrow}_{\leftarrow}$ in the subgraph $G^{s_k,t_k}_{x_k}$ is traversed from $x'_{k,s}$ to $\overline{x}'_{k,t}$ if $I(x_k) = 1$, and is traversed from $\overline{x}'_{k,t}$ to $x'_{k,s}$ if $I(x_k) = 0$. By Lemma 2, if $I(x_k) = 1$ then every γ-edge $(x_{k,l}, x'_{k,l})^{\rightarrow}_{\leftrightarrow}$ is traversed from $x_{k,l}$ to $x'_{k,l}$ and every γ-edge $(\overline{x}_{k,l}, \overline{x}'_{k,l})^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip, and if $I(x_k) = 0$ then every γ-edge $(x_{k,l}, x'_{k,l})^{\rightarrow}_{\leftrightarrow}$ is traversed as a round trip and every γ-edge $(\overline{x}_{k,l}, \overline{x}'_{k,l})^{\rightarrow}_{\leftrightarrow}$ is traversed from $\overline{x}_{k,l}$ to $\overline{x}'_{k,l}$. Additionally, by Lemma 2, each $G^{s_k,t_k}_{x_k}$ has no other valid edge tour.

We assume that F is satisfiable. Then, there exists a truth assignment I by which F is satisfied. In this case, at least one among c_{h1}, c_{h2} and c_{h3} is satisfied for every h ($1 \leq h \leq m$). Now, suppose that $c_{hh'}$ is satisfied under this assignment. If $c_{hh'} = x_k$ (or $\overline{x}_k$), then the corresponding γ-edge $(x_{k,l}, x'_{k,l})^{\rightarrow}_{\leftrightarrow}$ (or $(\overline{x}_{k,l}, \overline{x}'_{k,l})^{\rightarrow}_{\leftrightarrow}$), that is, $(p_{h,h'}, q_{h,h'})^{\rightarrow}_{\leftrightarrow}$ is traversed from $p_{h,h'}$ to $q_{h,h'}$ (We remark that $h = i_{k,l}$, $h' = j_{k,l}$). Edges in G_{C_h} can be traversed in the way described in the proof of Lemma 3. In addition, all δ-edges can be traversed as round trips. Such an edge tour constitutes a patrol route of G_F.

Conversely, suppose that G_F has a patrol route $\mathfrak{S}$. Then by the way of constructing the mixed graph G_F and Lemma 2, we can define a truth assignment $I_{\mathfrak{S}} : \{x_1, x_2, \cdots, x_n\} \rightarrow \{0, 1\}$ uniquely for the patrol route $\mathfrak{S}$ as follows:

(1) $I_{\mathfrak{S}}(x_k) = 1$ if every γ-edge $(p_{i_{k,l},j_{k,l}}, q_{i_{k,l},j_{k,l}})^{\rightarrow}_{\leftrightarrow}$ corresponding to s_k literals $c_{i_{k,1} j_{k,1}}, c_{i_{k,2} j_{k,2}}, \cdots, c_{i_{k,s_k} j_{k,s_k}}$ which are equal to x_k is traversed from $p_{i_{k,l},j_{k,l}}$ to $q_{i_{k,l},j_{k,l}}$ ($(x_{k,0}, x'_{k,0})^{\rightarrow}_{\leftrightarrow}$ is traversed from $x_{k,0}$ to $x'_{k,0}$ if $s_k = 0$).

(2) $I_{\mathfrak{S}}(x_k) = 0$ if every γ-edge $(p_{i'_{k,l},j'_{k,l}}, q_{i'_{k,l},j'_{k,l}})^{\rightarrow}_{\leftrightarrow}$ corresponding to t_k literals $c_{i'_{k,1} j'_{k,1}}, c_{i'_{k,2} j'_{k,2}}, \cdots, c_{i'_{k,t_k} j'_{k,t_k}}$ which are equal to $\overline{x}_k$ is traversed from $p_{i',j'}$ to $q_{i',j'}$ ($(\overline{x}_{k,0}, \overline{x}'_{k,0})^{\rightarrow}_{\leftrightarrow}$ is traversed from $\overline{x}_{k,0}$ to $\overline{x}'_{k,0}$ if $t_k = 0$).

Assume that for each i ($1 \leq i \leq m$), a γ-edge $(p_{i,j_i}, q_{i,j_i})^{\rightarrow}_{\leftrightarrow}$ ($1 \leq j_i \leq 3$) is traversed from p_{i,j_i} to q_{i,j_i} on the patrol route $\mathfrak{S}$. The existence of such γ-edge is guaranteed by Lemma 3. Let c_{ij_i} be the literal corresponding to the γ-edge $(p_{i,j_i}, q_{i,j_i})^{\rightarrow}_{\leftrightarrow}$. If the literal c_{ij_i} is equal to x_k then $I_{\mathfrak{S}}(c_{ij_i}) = I_{\mathfrak{S}}(x_k) = 1$. On the other hand, if the literal c_{ij_i} is equal to $\overline{x}_k$ then $I_{\mathfrak{S}}(c_{ij_i}) = I_{\mathfrak{S}}(\overline{x}_k) = 1$ since $I_{\mathfrak{S}}(x_k) = 0$. Hence, F is satisfiable since every clause $C_i = c_{i1} + c_{i2} + c_{i3}$ is satisfied by $I_{\mathfrak{S}}$. □

4. Concluding Remarks

We introduce an edge routing problem on mixed graphs representing urban areas and investigate its complexity. Our next interest is in how the computational complexity changes when POPP is restricted for instance, to undirected or directed graphs. We are especially interested in the complexity of POPP on undirected graphs since this is also a restricted variation of CVC. In addition, it would be interesting to study approximation algorithms for POPP by defining the problem on weighted graphs (where each street is given a distance).

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