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# Optimization Variant of Vertex-Coloring Reconfiguration Problem

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**Abstract:** Suppose that we are given a positive integer $k$, and a $k$-(vertex-)coloring $f_0$ of a given graph $G$. Then we are asked to find a coloring of $G$ using the minimum number of colors among colorings that are reachable from $f_0$ by iteratively changing a color assignment of exactly one vertex while maintaining the property of $k$-colorings. In this paper, we give linear-time algorithms to solve the problem for graphs of degeneracy at most two and for the case where $k \leq 3$. These results imply linear-time algorithms for series-parallel graphs and grid graphs. In addition, we give linear-time algorithms for chordal graphs and cographs. On the other hand, we show that, for any $k \geq 4$, this problem remains NP-hard for planar graphs with degeneracy three and maximum degree four. Thus, we obtain a complexity dichotomy for this problem with respect to degeneracy of a graph and the number $k$ of colors.

**Keywords:** Combinatorial reconfiguration, Graph algorithm, Graph coloring

## 1. Introduction

In *combinatorial reconfiguration*, we often consider the following problem: we are given two feasible solutions of a combinatorial search problem, then we are asked to determine whether one solution can be transformed into the other in a step-by-step fashion, such that each intermediate solution is also feasible. Such a problem is called *reconfiguration problem*. After Ito et al. proposed this framework [15], the reconfiguration problem has been extensively studied in the field of theoretical computer science. (See, e.g., the surveys of van den Heuvel [14] and Nishimura [22].)

Combinatorial reconfiguration models "dynamic" transformations of systems, where we wish to transform the current configuration of a system into a more desirable one by a step-by-step transformation. In the current framework of combinatorial reconfiguration, we need to have in advance a target (a more desirable) configuration. However, it is sometimes hard to decide a target configuration, because there may exist exponentially many desirable configurations. Based on this situation, Ito et al. introduced the new framework of reconfiguration problems, called the *optimization variant* [16].

In this variant, we are given a single solution as a current configuration, and asked for a more desirable solution reachable from the given one. This variant was introduced very recently, and hence it has only been applied to Independent Set Reconfiguration [16], [17] and Dominating Set Reconfiguration [1] to the best of our knowledge. Therefore, since Coloring Reconfiguration is one of the most studied reconfiguration problems [2], [3], [4], [5], [6], [8], [11], [13], [18], [24], we focus on this problem and study it under this framework.

### 1.1 Our problem

For an integer $k \geq 1$, let $C$ be a *color set* consisting of $k$ colors $1, 2, \ldots, k$. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Recall that a *$k$-coloring* $f$ of $G$ is a mapping $f \colon V(G) \to C$ such that $f(v) \neq f(w)$ holds for each edge $vw \in E(G)$.

In the (Vertex-)Coloring Reconfiguration problem, we are given two $k$-colorings $f_0$ and $f_r$ of the same graph $G$. Then we are asked to determine whether there is a sequence $\langle f_0, f_1, \ldots, f_\ell \rangle$ of $k$-colorings of $G$ such that $f_\ell = f_r$ and $f_i$ can be obtained from $f_{i-1}$ by recoloring only a single vertex in $G$ for all $i$, $1 \leq i \leq \ell$. Such a sequence is called *reconfiguration sequence* from $f_0$ to $f_r$. The Coloring Reconfiguration is one of the most studied reconfiguration problems [2], [3], [4], [5], [6], [8], [11], [13], [18], [24]. See also the survey of Mynhardt and Nasserasr [21].

In this paper, we study the *optimization variant* of Coloring Reconfiguration. We denote this problem by Opt-Coloring Reconfiguration. In Opt-Coloring Reconfiguration, we are given only one $k$-coloring $f_0$ of the given graph $G$. Then we are asked to find a $k$-coloring $f_{\mathsf{sol}}$ of $G$ such that there exists a reconfiguration sequence of $k$-colorings from $f_0$ to $f_{\mathsf{sol}}$, and $f_{\mathsf{sol}}$ uses the minimum number of colors over all colorings which can be transformed from $f_0$ through reconfiguration. We denote by $(G, k, f_0)$ an instance of Opt-Coloring Reconfiguration. Note that $f_{\mathsf{sol}}$ is not always a coloring of $G$ using the minimum number of colors among all colorings of $G$.

### 1.2 Related results

As we have mentioned above, Coloring Reconfiguration has

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been studied intensively.

For Coloring Reconfiguration, a sharp analysis under the number $k$ of colors has been obtained. It is known that Coloring Reconfiguration is PSPACE-complete for any fixed $k \geq 4$ [4]. On the other hand, it is known that Coloring Reconfiguration is solvable in linear time for any $k \leq 3$ [8], [18]. In addition, given a yes-instance of Coloring Reconfiguration for any $k \leq 3$, a reconfiguration sequence with shortest length can be found in polynomial time [8].

Coloring Reconfiguration has also been studied from the viewpoint of graph classes. It is known that Coloring Reconfiguration is PSPACE-complete for bipartite planar graphs [4]. Since every bipartite planar graph is 3-degenerate, Coloring Reconfiguration is PSPACE-complete for 3-degenerate graphs. Coloring Reconfiguration is known to be PSPACE-complete also for graphs with bounded bandwidth [24] and chordal graphs [13]. On the other hand, Coloring Reconfiguration is solvable in polynomial time for split, trivially perfect, 2-degenerate, and $(k-2)$-connected chordal graphs for any number $k$ of colors [6], [13].

The optimization variant of reconfiguration problems were recently proposed by Ito et al. [16]. To the best of our knowledge, it has only been applied to Independent Set Reconfiguration [16], [17] and Dominating Set Reconfiguration [1]. Therefore, in this paper, we apply this new framework to one of the most studied reconfiguration problems, namely Coloring Reconfiguration.

### 1.3 Our results

In this paper, we give linear-time algorithms to solve Opt-Coloring Reconfiguration for graphs of degeneracy two, and for any graph when $k \leq 3$. These results imply linear-time algorithms for series-parallel graphs and grid graphs. In addition, we give linear-time algorithms for chordal graphs and cographs for any $k$. Since Coloring Reconfiguration is PSPACE-hard for chordal graphs [13], we obtain a difference in complexity between Coloring Reconfiguration and Opt-Coloring Reconfiguration, that is, some difficulties disappear for the optimization variant, in a sense. On the other hand, we show that, for any $k \geq 4$, this problem remains NP-hard for planar graphs with degeneracy three and maximum degree four. Thus, we obtain a complexity dichotomy for this problem with respect to the number of colors and degeneracy of a graph.

## 2. Preliminaries

Let $G = (V, E)$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. We assume that all graphs in the remainder of this paper are simple, undirected, and have at least one edge. The *degeneracy* $d(G)$ of a graph $G$ is the minimum integer $d$ such that any subgraph $H$ of $G$ has a vertex of degree at most $d$. For a positive integer $k$, a graph $G$ is *$k$-colorable* if $G$ has a $k$-coloring. We say that a $k$-coloring $f$ of $G$ is *optimal* if $G$ has no $(k-1)$-coloring. We denote by $\chi(G)$, called the *chromatic number of* $G$, the integer $k$ such that $G$ has an optimal $k$-coloring.

A coloring $f$ of a graph $G$ is *$k$-reachable* from a coloring $f_0$ of $G$ if there is a sequence $\langle f_0, f_1, \ldots, f_\ell \rangle$ of $k$-colorings of $G$ such that $f_\ell = f$ and $f_i$ can be obtained from $f_{i-1}$ by recoloring only a single vertex of $G$ for every $i$, $1 \leq i \leq \ell$. For a coloring $f$ of $G$, let $\mathsf{col}(f)$ be the number of colors used in $f$. We define

$$\chi(G, k, f_0) = \min\{\mathsf{col}(f) \mid f \text{ is a coloring of } G \text{ and } f \text{ is } k\text{-reachable from } f_0\}$$

and $\chi(G, k, f_0) = +\infty$ if $k < \mathsf{col}(f_0)$. Note that $\chi(G, k, f_0)$ is at least $\chi(G)$. Opt-Coloring Reconfiguration is the problem of computing $\chi(G, k, f_0)$ for a given graph $G$, a positive integer $k$ and a coloring $f_0$ of $G$. We remark that, with minor adjustments, all algorithms in this paper can actually find a coloring $f_{\mathsf{sol}}$ of $G$ such that $\mathsf{col}(f_{\mathsf{sol}}) = \chi(G, k, f_0)$.

## 3. Linear-time algorithms

### 3.1 The case where the number of colors is at most three

In this subsection, we show the following theorem:

**Theorem 1.** *Let $(G, k, f_0)$ be an instance of* Opt-Coloring Reconfiguration. *If $k \leq 3$, the problem can be solved in linear time.*

*Proof.* Recall that the input graph $G$ has at least one edge. This implies that $\chi(G) > 1$ and thus $\chi(G, k, f_0) > 1$. If $f_0$ is a 2-coloring of $G$, then we conclude that $\chi(G, k, f_0) = 2$. In the remainder of this proof, we assume that $k = 3$ and hence $f_0$ is a 3-coloring of $G$.

We give an algorithm for an instance $(G, 3, f_0)$. Our algorithm contains the following two steps. First, the algorithm checks in linear time whether $G$ is 2-colorable, that is, bipartite. Since $f_0$ is a 3-coloring, $\chi(G)$ is two or three. If $G$ is not 2-colorable, we have $\chi(G) = 3$. In this case, the algorithm concludes that $\chi(G, k, f_0) = 3$, otherwise we go to the next step.

In the next step, the algorithm finds an arbitrary 2-coloring $f_r$ of $G$ in linear time, and then checks whether $f_r$ is 3-reachable from $f_0$ or not. It is known that Coloring Reconfiguration is solvable in linear time if $k \leq 3$ [18]. If $f_r$ is 3-reachable from $f_0$, the algorithm concludes that $\chi(G, k, f_0) = 2$, otherwise $\chi(G, k, f_0) = 3$. This step correctly outputs a solution because one can see that any 2-coloring is 3-reachable from any other 2-coloring. The total running time of our algorithm is linear, completing the proof. □

### 3.2 The graphs of degeneracy at most two

In this subsection, we show the following theorem:

**Theorem 2.** *Let $(G, k, f_0)$ be an instance of* Opt-Coloring Reconfiguration. *If the degeneracy $d(G)$ is at most two, then the problem can be solved in linear time.*

*Proof.* For the case where $k \leq 3$, we use the algorithm given in Theorem 1. Suppose that $k \geq 4$. It is known that, if $k \geq d(G) + 2$, then any two $k$-colorings of $G$ are $k$-reachable from each other [7]. Thus, for the case where $d(G) \leq 2$ and $k \geq 4$, we have $\chi(G, k, f_0) = \chi(G)$, and hence it suffices to compute $\chi(G)$. One can easily check whether or not $G$ is 2-colorable, that is, $\chi(G) = 2$ in linear time. If $\chi(G) \neq 2$, then $\chi(G) = 3$ because $d(G) \leq 2$ and $\chi(G) \leq d(G) + 1$. Thus, $\chi(G, k, f_0) = \chi(G)$ can be computed in linear time, completing the proof. □

Since both series-parallel and grid graphs have degeneracy at most two, we obtain the following corollary by Theorem 2:

**Corollary 1.** Opt-Coloring Reconfiguration *is solvable in linear time for series-parallel graphs and grid graphs.*

### 3.3 Chordal graphs

In this subsection, we show the following theorem:

**Theorem 3.** Opt-Coloring Reconfiguration *is solvable in linear time for chordal graphs.*

*Proof.* Let $(G, k, f_0)$ be an instance of Opt-Coloring Reconfiguration, where $G$ is a chordal graph. Suppose that $k \geq \mathsf{col}(f_0)$ holds. Our algorithm computes $\chi(G)$ and concludes that $\chi(G, k, f_0) = \chi(G)$. Since we can compute $\chi(G)$ in linear time for any chordal graph $G$ [23], our algorithm takes linear time.

We give the correctness of the algorithm. Clearly, if $\chi(G) = k$, then $f_0$ itself is an optimal coloring of $G$ and hence $\chi(G, k, f_0) = \chi(G)$ holds. We show that $\chi(G, k, f_0) = \chi(G)$ holds also for $\chi(G) < k$. It suffices to prove that any optimal coloring of $G$ is $k$-reachable from $f_0$ if $\chi(G) < k$. For any chordal graph $G$, $\chi(G) = d(G) + 1$ holds [20]. Thus, we have $k \geq d(G) + 2$. It is known that, if $k \geq d(G) + 2$, then any two $k$-colorings of $G$ are $k$-reachable [7]. Therefore, any optimal coloring of $G$ is $k$-reachable from $f_0$ if $\chi(G) < k$, and hence $\chi(G, k, f_0) = \chi(G)$, completing the proof. □

### 3.4 Cographs

In this subsection, we give a linear-time algorithm for cographs. In fact, the algorithm is almost the same as the one for chordal graphs. For the correctness of the algorithm, we use the Grundy number. A $k$-coloring $f_g$ of a graph $G$ is called a *Grundy coloring* if each vertex $v \in V(G)$ such that $f_g(v) = i$ is adjacent to at least one vertex with color $j$ for each $j < i$. The *Grundy number* $\chi_g(G)$ of $G$ is the maximum integer $k$ such that $G$ has a Grundy coloring with $k$ colors.

**Theorem 4.** Opt-Coloring Reconfiguration *is solvable in linear time for cographs.*

*Proof.* Let $(G, k, f_0)$ be an instance of Opt-Coloring Reconfiguration, where $G$ is a cograph. Suppose that $k \geq \mathsf{col}(f_0)$ holds. Our algorithm computes $\chi(G)$ and concludes that $\chi(G, k, f_0) = \chi(G)$. Since we can compute $\chi(G)$ in linear time for any cograph $G$ [23], our algorithm takes linear time.

We give the correctness of the algorithm. As in the proof of Theorem 3, we show that any optimal coloring of $G$ is $k$-reachable from $f_0$ if $\chi(G) < k$. For any cograph $G$, $\chi(G) = \chi_g(G)$ holds [9]. Thus, we have $k \geq \chi_g(G)+1$. It is known that, any two $k$-colorings of $G$ are $k$-reachable if $k \geq \chi_g(G) + 1$ [2]. Therefore, any optimal coloring of $G$ is $k$-reachable from $f_0$ if $\chi(G) < k$, and hence $\chi(G, k, f_0) = \chi(G)$, completing the proof. □

## 4. NP-hardness

In this section, we show that Opt-Coloring Reconfiguration remains NP-hard even for any $k \geq 4$, planar graphs with degeneracy three and maximum degree four. We assume that $k = 4$ because our proof can easily be applicable to the case where $k > 4$. Our proof consists of the following three steps:

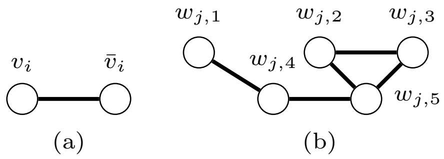


**Fig. 1** (a) A variable gadget $X_i$ and (b) a clause gadget $Y_j$.

**Step 1** construct an instance $(G_\phi, 4, f_\phi)$ of Opt-Coloring Reconfiguration from an instance $\phi$ of 3-SAT so that $G_\phi$ has degeneracy three;

**Step 2** transform $(G_\phi, 4, f_\phi)$ into $(G_p, 4, f_p)$ where $G_p$ is a planar graph of degeneracy three; and

**Step 3** reduce the maximum degree of the graph $G_p$ and construct an instance $(G, 4, f_0)$.

In 3-SAT, we are given a CNF-formula $\phi$ with a collection $\{C_1, C_2, \ldots, C_m\}$ of $m$ clauses over $n$ variables $\{x_1, x_2, \ldots, x_n\}$, and each clause contains exactly three variables. Our task is to determine whether there exists a variable assignment which satisfies a given CNF-formula or not. 3-SAT is a well-known NP-complete problem [19].

In fact, our construction of $G$ follows the existing reduction which proves the NP-hardness of 3-Coloring problem for planar graphs with degeneracy three and maximum degree four [10], [12]. Before we explain the construction of $G$ and $f_0$, we show that $\chi(G, 4, f_0) \geq 4$ if $\phi$ has no feasible variable assignment. In [10], [12], the authors proved that $G$ has a 3-coloring if and only if $\phi$ has a feasible variable assignment. Therefore, if $\phi$ has no feasible variable assignment, any coloring $f_0$ cannot reach any 3-coloring of $G$, and hence $\chi(G, 4, f_0) \geq 4$. Thus, in the remainder of this section, it suffices to give a 4-coloring $f_0$ of $G$ so that $\chi(G, 4, f_0) \leq 3$ if $\phi$ has a feasible variable assignment. In this technical report, we focus on only Step 1 and omit the explanation of Steps 2 and 3.

### 4.1 Step 1: constructing an instance from a CNF-formula

As the first step in our reduction, we explain how to construct an instance $(G_\phi, 4, f_\phi)$ of Opt-Coloring Reconfiguration from an instance $\phi$ of 3-SAT, where $G_\phi$ has degeneracy three. In the construction, we use a variable gadget and a clause gadget in Fig. 1, which appears in [10]. The variable gadget $X_i$, $1 \leq i \leq n$, consists of two vertices $v_i$ and $\bar{v}_i$. The clause gadget $Y_j$, $1 \leq j \leq m$, consists of five vertices $w_{j,1}, w_{j,2}, \ldots, w_{j,5}$.

For a given CNF-formula $\phi$, we build a corresponding graph $G_\phi$ as follows. First, for each variable $x_i$ with $1 \leq i \leq n$ and each clause $C_j$ with $1 \leq j \leq m$ of $\phi$, we add one variable gadget $X_i$ and one clause gadget $Y_j$, respectively. We also add a cycle of three vertices $v_T, v_F$ and $v_B$. We connect $v_B$ to $v_i$ and $\bar{v}_i$ in each variable gadget $X_i$ by edges, and connect $v_T$ to $w_{j,1}$ and $w_{j,4}$ in each clause gadget $Y_j$ by edges. Then, if a variable $x_i$ (resp. $\bar{x}_i$) appears at the $\ell$-th position of a clause $C_j$ of $\phi$, we connect $v_i$ (resp. $\bar{v}_i$) of the variable gadget $X_i$ and $w_{j,\ell}$ of the clause gadget $Y_j$ by an edge, as illustrated in Fig. 2. This completes the corresponding graph $G_\phi$. Clearly, $G_\phi$ is constructed in polynomial time. From the construction of $G_\phi$, it is not hard to see that $G_\phi$ has degeneracy three.

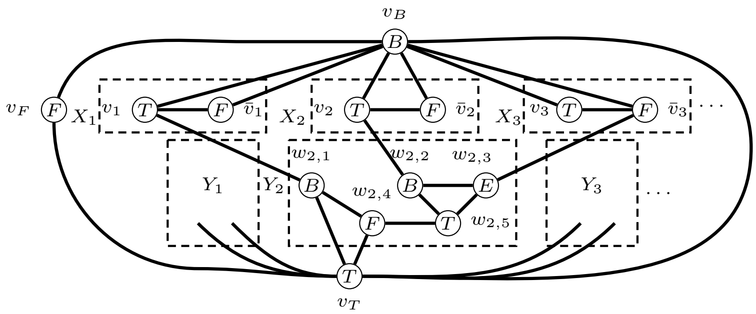


**Fig. 2** An example of the construction of a graph $G_\phi$ and a 4-coloring $f_\phi$ of $G_\phi$, where $C_2 = x_1 \vee x_2 \vee \bar{x}_3$.

Next we explain the construction of $f_\phi$. Let $\{T, F, B, E\}$ be a color set. The vertices $v_T, v_F$ and $v_B$ are colored by $T, F$ and $B$, respectively. For each variable gadget $X_i$, $1 \le i \le n$, $v_i$ is colored by $T$ and $\bar{v}_i$ is colored by $F$. For each clause gadget $Y_j$, $1 \le j \le m$, $w_{j,1}$ and $w_{j,2}$ are colored by $B$, $w_{j,3}$ is colored by $E$, $w_{j,4}$ is colored by $F$ and $w_{j,5}$ is colored by $T$. Clearly, our construction of $f_\phi$ is done in polynomial time. Then, we have the following lemma, whose proof is omitted in this technical report.

**Lemma 1.** *$\chi(G_\phi, 4, f_\phi) \le 3$ if $\phi$ has a feasible variable assignment.*

## 5. Conclusion

In this paper, we gave linear-time algorithms to solve the problem for graphs of degeneracy at most two and for the case where $k \le 3$. These results imply linear-time algorithms for series-parallel graphs and grid graphs. In addition, we gave linear-time algorithms for chordal graphs and cographs. On the other hand, we showed that, for any $k \ge 4$, this problem remains NP-hard for planar graphs with degeneracy three and maximum degree four. In particular, our theorems give a sharp complexity dichotomies with respect to the degeneracy of the input graph and the number $k$ of colors.

It remains open to clarify the complexity status of perfect graphs, bipartite graphs, or graphs of maximum degree three.

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