Rep-cube: Research on dissections of a net of a cube into nets of cubes

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Abstract: A rep-cube is a polyomino that is a net of a cube, and it can be divided into some polyominoes such that each of them can be folded into a cube. This notion was invented in 2017, which is inspired by the notions of polyomino and rep-tile, which were introduced by Solomon W. Golomb. A rep-cube is called regular if it can be divided into the nets of the same area. A regular rep-cube is of order k if it is divided into k nets. Moreover, it is called uniform if it can be divided into the congruent nets. In this paper, we focus on these special rep-cubes and solve several open problems.

Keywords: Computational origami, polyomino, rep-cube, rep-tile

1. Introduction

A *polyomino* is a "simply connected" set of unit squares introduced by Solomon W. Golomb in 1954 [7]. Since then, sets of polyomino pieces have been playing an important role in recreational mathematics (see, e.g., [5]). In 1962, Golomb also proposed an interesting notion called *rep-tile*: a polygon is a rep-tile of order k if it can be divided into k replicas congruent to one another and similar to the original (see [6], Chap 19). From these notions, Abel et al. introduced a new notion [1]; a polyomino is said to be a *rep-cube* of order k if it is a net of a cube (or, it can fold into a cube), and it can be divided into k polyominoes of which each can fold into a cube. If all k polyominoes have the same size, we call the original polyomino a *regular* rep-cube of order k. Moreover, a regular rep-cube is a *uniform* rep-cube of order k when all k polyominoes are congruent. Simple examples of a regular rep-cube and a uniform rep-cube are shown in Fig. 1(a) and (b), respectively. We note that crease lines are not necessarily along the edges of the polyomino as shown in the figure.

In [1], Abel et al. showed concrete regular rep-cubes of order k for $k = 2, 4, 5, 8, 9, 36, 50, 64$. Later, in [16], Xu et al. also gave regular rep-cubes of order $k = 16, 18, 25$. In both papers, they showed some ways of construction of regular rep-cubes of order k for infinitely many integers k. In these papers, the following two sets play important roles;

$$
\begin{aligned}
S &= \{k \mid a^2 + b^2 = k \text{ for two integers } a, b\} \\
\bar{S} &= \mathcal{Z} \setminus S
\end{aligned}
$$

namely, $S = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, \ldots\}$ and $\bar{S} = \{3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, \ldots\}$.

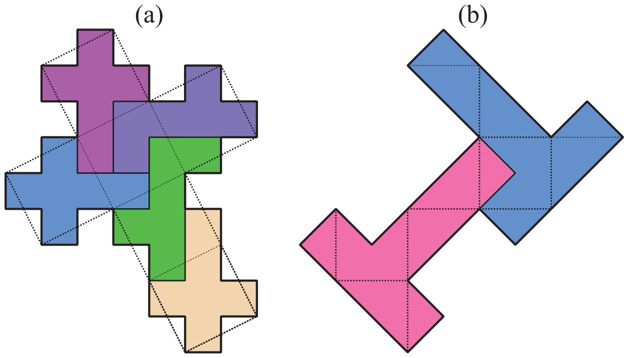
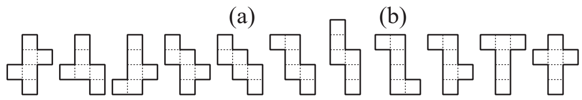


Fig. 1 (a) A regular rep-cube of order 5 and (b) a uniform rep-cube of order 2.

We can observe that all the integers where there exists a regular rep-cube of order k are in S. We note that both of S and $\bar{S}$ are infinite sets by Dirichlet's theorem on arithmetic progressions.



k=2	**N**	**Y**	**Y**	**N**	**N**	**N**	**Y**	**N**	**N**	**Y**	**N**
k=4	**Y**	**Y**	**Y**	**Y**	**N**	**Y**	**N**	**N**	**Y**	**Y**	**Y**
k=5					**N**			**N**			
k=8					**Y**			**Y**			

Fig. 2 Eleven nets obtained by cutting along edges of a cube and their minimum number of copies to cover a cube.

On the other hand, in [16], they showed that there are no regular rep-cube of order 3. They proved that if $k \in \bar{S}$, there does not exist a regular rep-cube of area $6k$ of order k. Intuitively speaking, they showed that k copies of one net in Fig. 2 cannot cover a cube of area $6k$ if k is in $\bar{S}$. However, they could not prove that it holds for general regular rep-cubes of order k in $\bar{S}$. We first solve this open problem. That is, we prove that there does not

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exist a regular rep-cube of order k if k is in $\bar{S}$. In other words, any set of k (refined) polyominoes of the same area cannot cover a cube of area $6k$ if k is in $\bar{S}$. (In [16], this claim was proved only for $k = 3$.) Oppositely, we conjecture that there exists a regular rep-cube of order k if $k \in S$; however, we have to construct one by one so far. In this paper, we give regular rep-cubes of order $k = 10, 13, 17, 20$, which did not appear in [1], [16].

Next we focus on uniform rep-cubes of order k, which consist of k copies of congruent nets. As a net of a cube, the eleven nets shown in Fig. 2 are quite popular since they are obtained by cutting along edges of a cube. (In the context of unfolding, they are sometimes called *edge-unfolding* of a cube.) Moreover, through the enumeration of regular rep-cubes of order $k = 2$ and $k = 4$ in [16], we can observe that nine of eleven nets form uniform rep-cubes. That is, for example, two copies of a net of T shape shown in Fig. 1(b) cover a cube. In this context, it is natural to ask how many copies we need to cover a cube by each of eleven nets. Especially, can the last remaining two nets, indicated (a) and (b) in Fig. 2, form uniform rep-cubes? Our second results state that for both of two nets, we can cover a cube by eight copies of them, and we cannot cover by five copies as shown in Fig. 2.

Lastly, we consider a new notion of a *universal* rep-cube that contains all of eleven nets in Fig. 2. This notion itself first proposed in [1] as an example of a regular rep-cube of order $k = 50$ with no special name. In [16], the authors showed another one with $k = 25$. Trivially, k is greater than or equal to eleven, and k should be in S. Thus the minimum number of the universal rep-cube of order k is $k = 13, 16, 17, 18, 20$, or 25. In this paper, we prove that $k = 13$, which solves the open problem shown in [16]. In this context, Maekawa proposed an interesting puzzle for this problem [9]: We consider two polygons are different if they are mirror images with each other. The set of eleven nets of a unit cube contains two mirror symmetric shapes (T-shape and +-shape, which appear in the right most two in Fig. 2). Let S be the set of nets of a unit cube, where mirror images are different with each other. Then S consists of 20 nets, and hence the nets in S are of area 120 in total. The puzzle asks if you can make a rep-cube of area 120, or a cube of size $2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$ from this set S without flipping each net. We give an affirmative answer to this problem. That is, there is a universal rep-cube that uses 20 different nets exactly once for each. In order to find these large rep-cubes, we use SCIP [17], which is one of the fastest non-commercial solvers for mixed integer programming.

2. Nonexistence of regular rep-cubes

The main theorem in this section is as follows.

Theorem 1 There does not exist a regular rep-cube of order k for each $k \in \bar{S}$.

In order to show it, we use the following theorem, which is a folklore in puzzle society (see [16]):

Theorem 2 (1) Let p be a prime. Then p can be represented by $p = a^2 + b^2$ for some two nonnegative integers a and b if and only if either $p = 2$ (with $a = b = 1$) or $p \equiv 1 \pmod 4$. (2) Let x be a composite number. Let $p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}$ be the prime factorization of x. Then x is can be represented by $x = a^2 + b^2$ for some two nonnegative integers a and b if and only if d_i is even for every prime p_i with $p_i \equiv 3 \pmod 4$.

Theorem 2(1) is known as "Fermat's theorem on sums of two squares," which was proposed by Fermat, and first proof was found by Euler.

Now we give the proof of Theorem 1:

Proof. We prove the claim by a contradiction. We assume that $\hat{P}$ is a regular rep-cube of order k. Then $\hat{P}$ can be divided into k nets $P_1, \ldots, P_k$. Let $\hat{Q}$ be a cube folded from $\hat{P}$ and Q a cube folded from P_i for each $i = 1, \ldots, k$. Let ℓ be the length of an edge of Q. Then P_i is a $6\ell^2$-omino and $\hat{P}$ is a $6k\ell^2$-omino. Here, we note that while $6\ell^2$ is an integer, ℓ is not necessarily an integer.

Now, using the same argument in [16], we can put P_i on a square lattice of size ℓ so that every vertex of Q is on a grid point. In other words, there are some positive integers a, b such that $a^2 + b^2 = \ell^2$. Using the same argument for $\hat{P}$ and $\hat{Q}$, we obtain $\hat{a}^2 + \hat{b}^2 = k\ell^2$ for some positive integers $\hat{a}, \hat{b}$. Therefore, $k\ell^2$ is an element in S, and we have

$$\hat{a}^2 + \hat{b}^2 = k\ell^2 = k(a^2 + b^2).$$

That is, a composite number $k(a^2+b^2)$ is in S. On the other hand, k is in $\bar{S}$ by assumption. Thus, when k is a prime, we have $k \equiv 3 \pmod 4$. When k is a composite number, its prime factorization contains a prime p_i such that $p_i \equiv 3 \pmod 4$ and its degree d_i is an odd number. We can regard the first case (k is a prime) as the special case of the second case with $p_1 = k$ and $d_1 = 1$ with no other factors. Thus we focus on the second case.

Now, a composite number $k(a^2 + b^2) = \hat{a}^2 + \hat{b}^2$ is in S. Therefore, the factor $(a^2 + b^2)$ should contain p_i odd times as factors, which contradicts the fact that $(a^2 + b^2)$ is an element in S.

Therefore, there exists no such k, and hence there exists no regular rep-cube of order k. ■

3. Minimum uniform rep-cubes

In the previous work, there exists uniform rep-cubes of order k for each $k = 2, 4, 9$ in [1]. In [16], it is shown that how to construct infinitely many uniform rep-cubes recursively. Summarizing known uniform rep-cubes in Fig. 2, it is natural to ask if remaining two of eleven nets can form uniform rep-cubes or not. Let name the nets Fig. 2(a) and Fig. 2(b) P_w and P_z (from their shapes), respectively. For these two nets, we show the following theorem.

Theorem 3 Using P_w and P_z, we can construct uniform rep-cubes of order k for $k = 8$. Moreover, we cannot construct uniform rep-cubes of order k with $k = 2, 4, 5$.

We have the following corollary.

Corollary 4 For each one of eleven nets in Fig. 2, k copies of one can cover a cube for some $k = 2, 4, 5$, or 8.

By enumerations in [16] for $k = 2, 4$, Thoerem 3 and Corollary 4 hold except P_w and P_z. Thus we focus on P_w and P_z.

Lemma 5 There exist uniform rep-cubes of order 8 by P_w and P_z.

Proof. We prove the claim by construction. See Fig. 3. ■

Next we show the following lemma.

Lemma 6 There does not exist a uniform rep-cube of order 5 by P_w or P_z.

Proof. The proof is done by case analysis. We first focus on P_w.

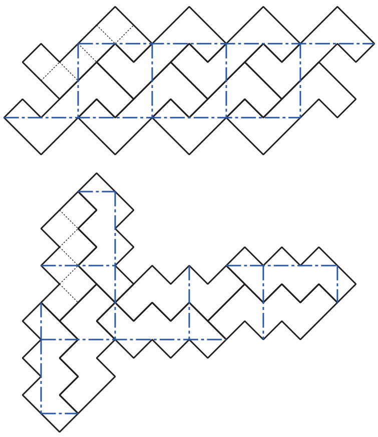

Fig. 3 Minimum uniform rep-cubes by P_w and P_z.

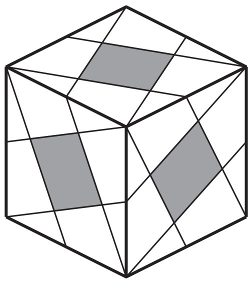

Fig. 4 A cube Q of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$.

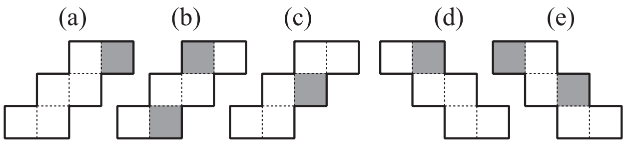

Fig. 5 Five possible ways for P_w.

If five copies of P_w form a uniform rep-cube of order 5, the resulting cube Q is depicted in Fig. 4 (or its mirror image). The cube Q has six *central squares* at each of six faces as shown in gray in Fig. 4. Then it is not difficult to see that P_w can put on Q with respect to the central squares in 5 different ways as shown in Fig. 5. To derive contradictions, we assume that five copies of P_w can cover on Q without any overlapping and any hole. Then we have two cases. The first case is that one copy of P_w is in the case Fig. 5(b) and the other four copies are in the cases Fig. 5(a)(c)(d), and the second case is that one copy of P_w is in the case Fig. 5(e) and the others are in the cases Fig. 5(a)(c)(d).

We first consider the first case; that is, a copy P_1 of P_w is in the case Fig. 5(b). Then, beside P_1, we have two central squares to be covered. We pick up one of them and consider how we can cover it by a copy P_2 of P_w. Then P_2 is in the case Fig. 5(a)(c) or (d). For each of them, we have four ways of orientation of P_2. Then, in most cases, (1) P_1 and P_2 surround a unit square or a small rectangle of two unit squares, or (2) P_2 overlaps with P_1. The only exception occurs one orientation in the case Fig. 5(a). Thus we have only one way of attaching P_2 beside P_1. Now we consider the next copy P_3 of P_w which covers the other neighbor central square of ones covered by P_1. Then we can use the same argument, and we have one way of attaching P_3 on Q. Then, we can find that P_3 overlaps P_2. Thus, in this case, we have no way to cover Q by five copies of P_w.

Next, we consider the second case; that is, a copy P_1 of P_w is in the case Fig. 5(e). Then, beside P_1, we have two central squares to be covered again. Then we can use the same argument of the first case. We consider all ways of attaching of two neighbor central squares of P_1, and we have the same conclusion; we have an overlap or a hole when we attach P_1, P_2, and P_3.

Thus, there does not exist a uniform rep-cube of order 5 by P_w.

For the P_z, we can have the similar case analysis, and confirm that there does not exist a uniform rep-cube of order 5 by P_z. (We note that the number of cases increases because P_z can attach on Q without covering any central square, however, the arguments are essentially the same, and hence omitted here.) ■

By Lemmas 5 and 6 with known enumeration in [16], Theorem 3 immediately follows.

4. Universal rep-cubes

We say that a regular rep-cube of order k is *universal* if it can be divided into k polyominoes in Fig. 2 such that the set contains all of eleven nets. This notion was introduced in [1] without name, and it was shown for $k = 50$. Later, it was improved to $k = 25$ as shown in [16]. It is a natural question for finding the minimum k such that a universal rep-cube exists. In this section, we prove that $k = 13$ by construction.

Theorem 7 The minimum number k such that there exists a universal rep-cube is $k = 13$.

Proof. By Theorem 1 and known result in [16], we can observe that $k = 13, 16, 17, 18, 20$, or 25. Since there is a universal rep-cube of order 13 as shown in Fig. 6, we have the claim. ■

In this context, Maekawa proposed an interesting puzzle for this problem [9]: We consider two polygons are different if they are mirror images with each other. The set of eleven nets of a unit cube contains two mirror symmetric shapes (T-shape and +-shape, which are rightmost in Fig. 2). Here, let S be the set of the nets of a unit cube, where mirror images are regarded as different with each other. Then S consists of 20 nets, and hence the nets in S are of area 120 in total. The Maekawa's puzzle asks if you can make a rep-cube of area 120, or a cube of size $2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$ from this set S without flipping each net. We give an affirmative answer to this problem by constuction.

Theorem 8 There exists a universal rep-cube of order $k = 20$ such that every different net (with respect to flip) appears exactly once.

Proof. A solution is shown in Fig. 7. ■

The reader may wonder how we can find them. In fact, one of the authors found the pattern of a universal rep-cube of order $k = 25$ in [16] by his hand and it was a really puzzle. We found the patterns in Fig. 6 and Fig. 7 by using SCIP [17], which is one of the fastest non-commercial solvers for mixed integer programming. We give the formulation of our problem for solving by SCIP.

4.1 Integer Programming Formulation

We formulate the problem in terms of a 0-1 integer programming problem. Although we found the patterns in Fig. 3 by our hands and prove Lemma 6 by case analysis, we use the case $k = 5$

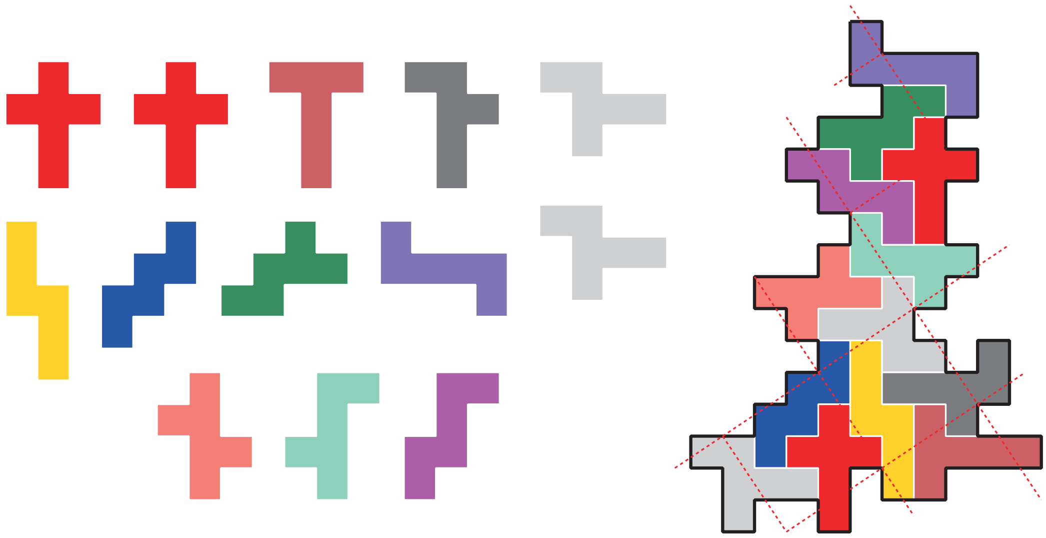

Fig. 6 A minimum universal rep-cube of order $k = 13$.

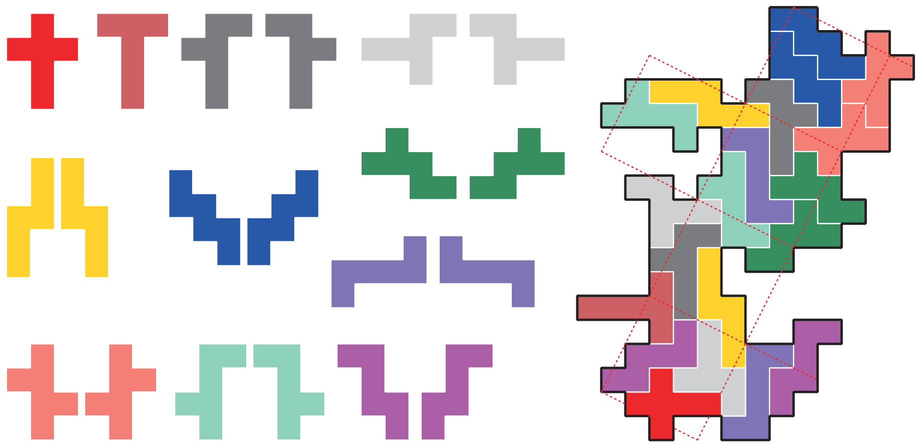

Fig. 7 A soluton of Maekawa's puzzle.

for explanation.

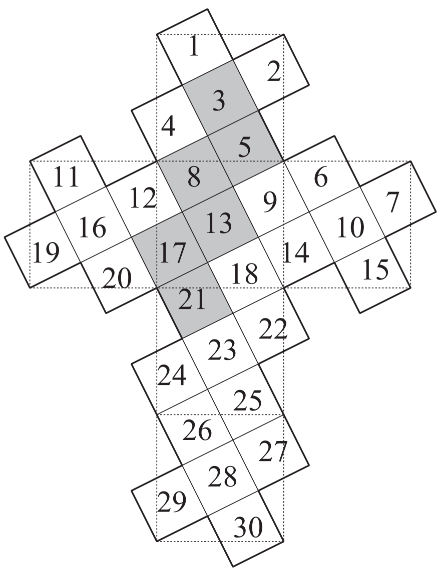

Fig. 8 Numbering the unit squares on a cube.

We first number all unit squares on the target cube Q (see Fig. 8 for a cube of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$; the ordering is arbitrary). We name each square i for each $i = 1, 2, \ldots, 30$ for reference. Then, for each placement of P_w, we use a 0-1 integer variable. In Fig. 8, a placement of P_w is indicated in gray. For this position, we define a 0-1 integer variable $P_w(3, 5, 8, 13, 17, 21)$. For each possible placement, we prepare one 0-1 integer variable $P_w(i_1, i_2, i_3, i_4, i_5, i_6)$, where i_j indicates the name of the corresponding unit square. For each unit square i, there are four copies of P_w that contain i at the end of P_w. We have to consider the mirror image of P_w in this case. We denote it by $P^r_w(i_1, i_2, i_3, i_4, i_5, i_6)$. Therefore, we have eight variables for each unit square i that consist of four $P_w(i_1, i_2, i_3, i_4, i_5, i_6)$s and four $P^r_w(i_1, i_2, i_3, i_4, i_5, i_6)$s such that each of them contains the square i at the end. However, we have duplicates; for example, two variables $P_w(3, 5, 8, 13, 17, 21)$ and $P_w(21, 17, 13, 8, 5, 3)$ are essentially the same. Thus we define the standard form that $i_1 < i_6$ for $P_w(i_1, i_2, i_3, i_4, i_5, i_6)$ and we only use the variables of the standard form. Therefore, we have $30 \times 4 \times 2/2 = 120$ 0-1 integer variables for this case.

Now we consider the constraints. For each square i, it should be covered by exactly once by a copy of

P_w. In order to represent it, we have the following constraint for each i: $\sum_{i \in P_w(i_1,i_2,i_3,i_4,i_5,i_6)} P_w(i_1,i_2,i_3,i_4,i_5,i_6) + \sum_{i \in P^r_w(i_1,i_2,i_3,i_4,i_5,i_6)} P^r_w(i_1,i_2,i_3,i_4,i_5,i_6) = 1$. In total, we have 30 constraints.

The objective function is simply given by minimize $\sum(P_w(i_1,i_2,i_3,i_4,i_5,i_6) + P^r_w(i_1,i_2,i_3,i_4,i_5,i_6))$. The solution should be 5 in this case since we use five copies of P_w or P^r_w to cover the cube Q.

In fact, the proof of Lemma 6 was double-checked by SCIP, and it confirmed that there is no solution for this case in 0.00 second. (We use SCIP version 7.0.0 on a laptop PC (AMD Ryzen 7, 2.30GHz, 16GB RAM, 64bit Windows).)

For finding the pattern in Fig. 6, we prepare 4960 variables for representing positions and 78 constraints for unit squares. We add eleven constraints so that each of eleven net appears at least once. In this case, the pattern in Fig. 6 was found in 17.00 seconds.

For finding the pattern in Fig. 7, we prepare 7680 variables and 140 constraints. Among them, 120 constraints are for unit squares and additional 20 constraints represent that each of 20 nets appears exactly once. SCIP found the pattern in Fig. 7 in 982.00 seconds.

5. Concluding Remarks

In this paper, we investigated uniform rep-cubes and universal rep-cubes. In general, we characterized the numbers that a regular rep-cube of order k can exist if k is in S, where $S = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, \ldots\}$ and $\bar{S} = \{3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, \ldots\}$. Precisely, we can say that if k is in $\bar{S}$, we cannot find a regular rep-cube of order k. Even if k is in S, we have no idea whether it exists or not without explicit construction. In [1], [16], they explicitly gave a regular rep-cube of order k for $k = (1), 2, 4, 5, 8, 9, 16, 18, 25, 36, 50, 64$. In this paper, we gave $k = 13$ (Fig. 6) and $k = 20$ (Fig. 7).

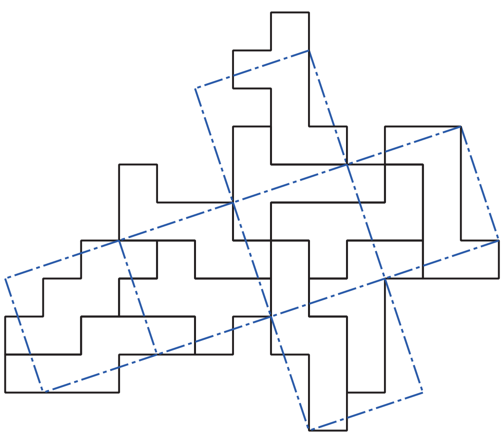

Fig. 9 A regular rep-cube of order $k = 10$.

For $k = 10$, we found by hand as shown in Fig. 9. On the other hand, for $k = 17$, we use the same way for finding a universal rep-cube of order 13 in Fig. 6. In this case, we have 6528 0-1 integer variables with 124 constraints, and SCIP found the solution shown in Fig. 10 in 11.00 seconds. In summary, we found regular rep-cubes of order k with all possible $k \in S$ with $k \leq 25$. It seems that there exists a regular rep-cube of order k for any $k \in S$. That is an open question.

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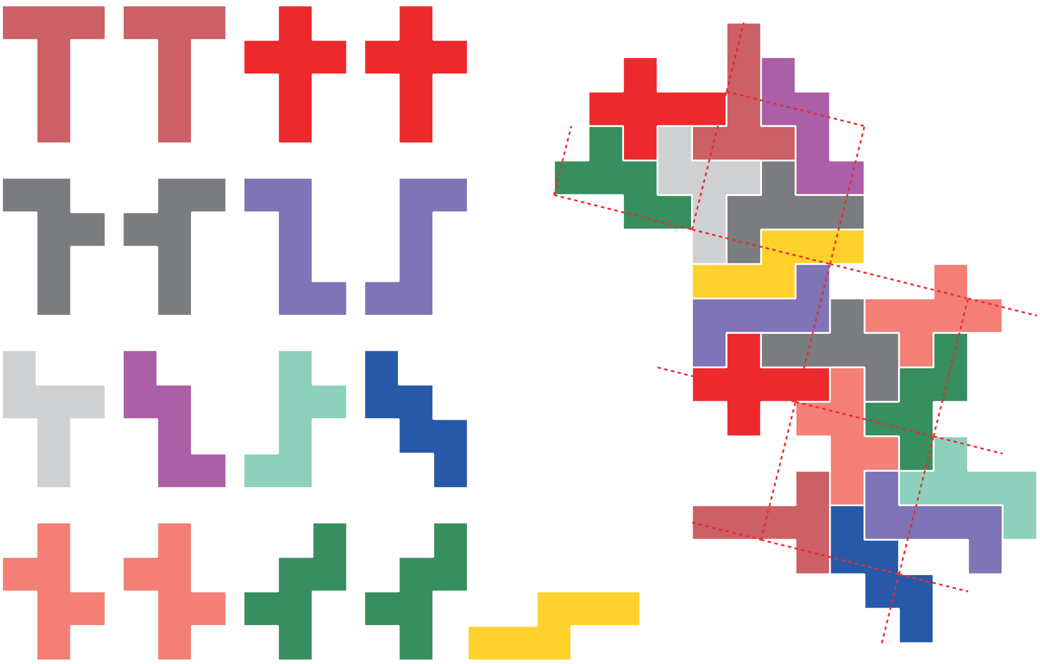

Fig. 10 A universal rep-cube of order $k = 17$.

