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Technical Note

Rectangular Unfoldings of Polycubes

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Abstract: In this paper, we investigate the problem that asks if there exists a net of a polycube that is exactly a rectangle with slits. For this nontrivial question, we show affirmative solutions. First, we show some concrete examples: (1) no rectangle with slits with fewer than 24 squares can fold to any polycube, (2) a 4×7 rectangle with slits can fold to a heptacube (nonmanifold), (3) both of a 3×8 rectangle and a 4×6 rectangle can fold to a hexacube (nonmanifold), and (4) a 5×6 rectangle can fold to a heptacube (manifold). Second, we show a construction of an infinite family of polycubes folded from a rectangle with slits. The smallest one given by this construction is a 6×20 rectangle with slits that can fold to a polycube of genus 5. This construction gives us a polycube for any positive genus. Moreover, by this construction, we can show that there exists a rectangle with slits that can fold to k different polycubes for any given positive integer k.

Keywords: computational origami, polycube, unfolding

1. Introduction

It is well known that a unit cube has eleven edge developments. When we unfold the cube, no overlap occurs on any of these eleven developments. In fact, any development of a regular tetrahedron is a tiling, and hence no overlap occurs [1]. However, this is not necessarily true for a general polycube, which is a polyhedron obtained by face-to-face gluing of unit cubes. For example, we can have an overlap when we unfold a box of size $1 \times 1 \times 3$ (**Fig. 1**), while we have no overlap when we unfold a box of size $1 \times 1 \times 2$ (checked by exhaustive search). On the other hand, even for the Dali cross (3-dimensional development of 4-dimensional hyper cube), there is a non-overlapping unfolding that is a polyomino with slits that satisfies Conway's criterion in the induced plane tiling [2].

In this context, we investigate a natural but nontrivial question that asks if we can fold a polycube from a rectangle with slits or not. We first note that a convex polycube (or a "box") cannot be folded from any rectangle with slits. In general, slits are irrelevant for a development of a convex polycube as proved in Ref. [3], Lemma 1. Therefore, a rectangle cannot fold to any convex polycube even if we make slits in any way. That is, if the answer to the question is yes, the polycube should be concave.

In this paper, we show two series of affirmative answers to the question. First, we use a computer program that searches slits of a given rectangle to fold a polycube. Based on the algorithm, we find some concrete slit patterns:

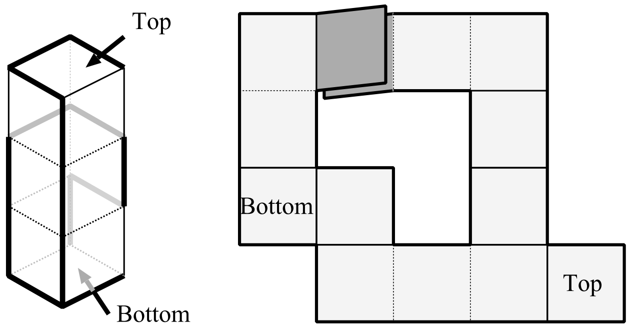

Fig. 1 Cutting along the bold lines of the left box of size $1 \times 1 \times 3$, overlap occurs at the dark gray square on the right development. This development was first found by Takeaki Uno in 2008. We have four places to glue the top; however, this development is essentially unique way for this box to overlap except the place of the top, which is examined by exhaustive search.

Theorem 1 (1) No rectangle with slits with fewer than 24 squares can fold to any polycube. (2) A 4×7 rectangle with slits can fold to a nonmanifold heptacube. (3) Both of a 3×8 rectangle and a 4×6 rectangle can fold to a nonmanifold hexacube. (4) A 5×6 rectangle can fold to a manifold heptacube.

Second, we show a construction of an infinite family of polycubes folded from a rectangle with slits.

Theorem 2 For any positive integer g, there is a rectangle with slits that can fold to a polycube of genus g.

As a result, we can conclude that there are infinite many polycubes that can be folded from a rectangle with slits. In the construction in Theorem 2, we use a series of gadgets that can be folded in many different ways of folding. Using this property, we also have the following corollary.

Corollary 3 For any positive integer k, there is a rectangle

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with slits that can fold to at least k different polycubes.

In this paper, we concentrate on folding of rectangular grid. That is, each sheet is a rectangle that consists of unit squares with some slits on the grid lines. The sheet is one piece, or it is not disconnected by the slits. We fold only along on grid lines.

2. Proof of Theorem 1

We first show the results and give a brief idea of the algorithm that we used for finding the patterns. In **Fig. 2**, **3**, and **4**, D, B, R, L, U, F in the unfolding mean Down, Back, Right, Left, Up, Front, respectively.

2.1 Pattern 1: 4 × 7 Rectangular Unfolding of a Heptacube

In Fig. 2, we give a heptacube that has a 4×7 rectangular unfolding. Cubes a and f touch along a diagonal. This heptacube has 90 rectangular unfoldings.

2.2 Patterns 2 and 3: 3×8 and 4×6 Rectangular Unfoldings of a Symmetric Hexacube of Genus 1

In Fig. 3, we give a hexacube that has two rectangular unfoldings. One is of size 3×8 and the other is of size 4×6. This polycube has no pair of unit cubes sharing an edge (like a and f in Fig. 2) although it is genus 1 at the central point. There are 1440 rectangular unfoldings, and two representative rectangular unfoldings are shown in Fig. 3. (This 24-face hexacube has 12 symmetries; therefore, the number of distinct rectangular unfoldings is 120 rather than 1440.)

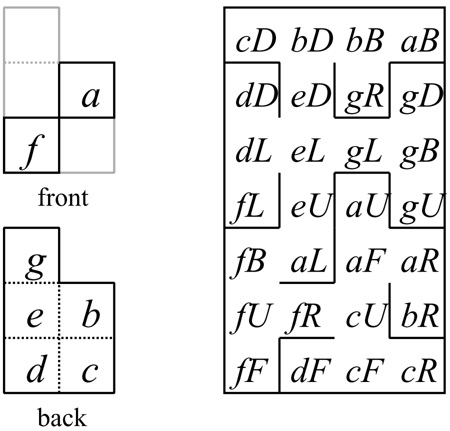

Fig. 2 The right rectangle is an unfolding of the left polycube of volume 7. The left figures specify the polycube by its cross-sections.

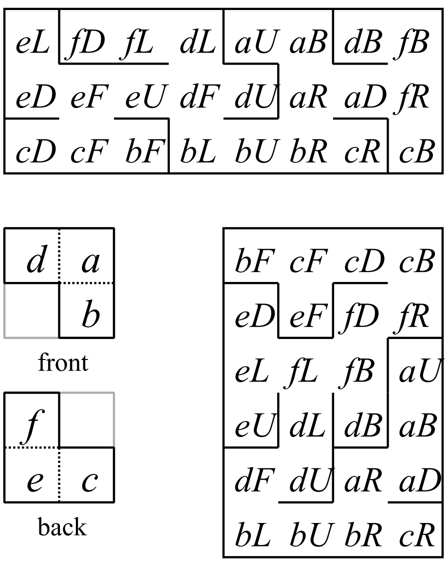

Fig. 3 The left down polycube of volume 6 has two different rectangular unfoldings of size 3×8 and 4×6 with slits.

2.3 Pattern 4: 5 × 6 Rectangular Unfolding of a Symmetric Heptacube

In Fig. 3, we give a heptacube that has rectangular unfolding of size 5×6. This polycube has no diagonal touch with genus 0. Curiously, this heptacube has only 4 rectangular unfoldings. All unfoldings are shown in Fig. 4. As you can observe, these 4 unfoldings are almost the same except the cut of unit length at the top left corner.

Our program confirmed that there are no rectangular polycube unfoldings with fewer than 24 faces, and the one shown in Fig. 3 is unique for 24 faces. These facts complete the proof of Theorem 1. We give the outline of the algorithm used in this section.

2.4 Algorithm

The input of our algorithm is a polycube Q. We here consider the polycube Q of surface area n squares as a graph $G(Q) = (V, E)$; the set V of n unit squares and $E = \{\{u, v\} \mid$ the unit squares u and v share an edge on $Q\}$. On the graph $G(Q)$, a slit on Q cuts the corresponding edge. Then it is known that an unfolding of the polycube Q is given by a spanning tree of $G(Q)$ (see, e.g., Ref. [3]). For example, the cutting pattern in Fig. 1 corresponds to the spanning tree in the right graph in **Fig. 5**. There-

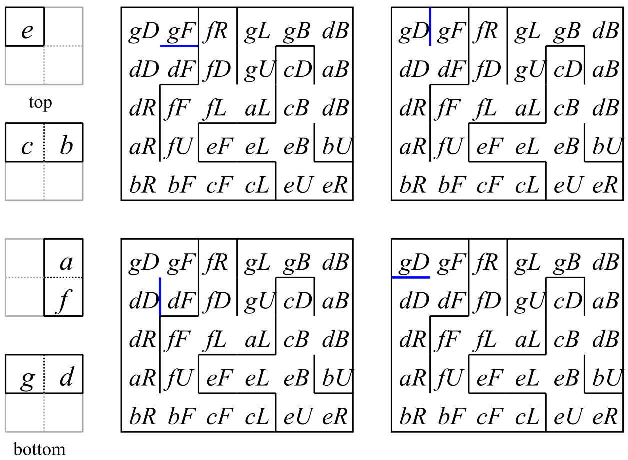

Fig. 4 The left polycube of volume 7 has rectangular unfolding of size 5×6.

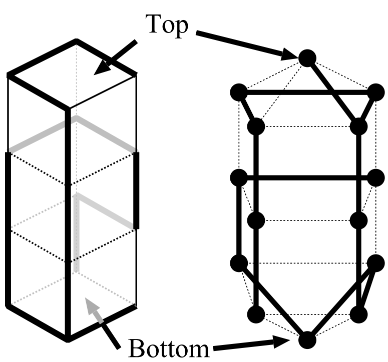

Fig. 5 Spanning tree corresponding to the pattern in Fig. 1.

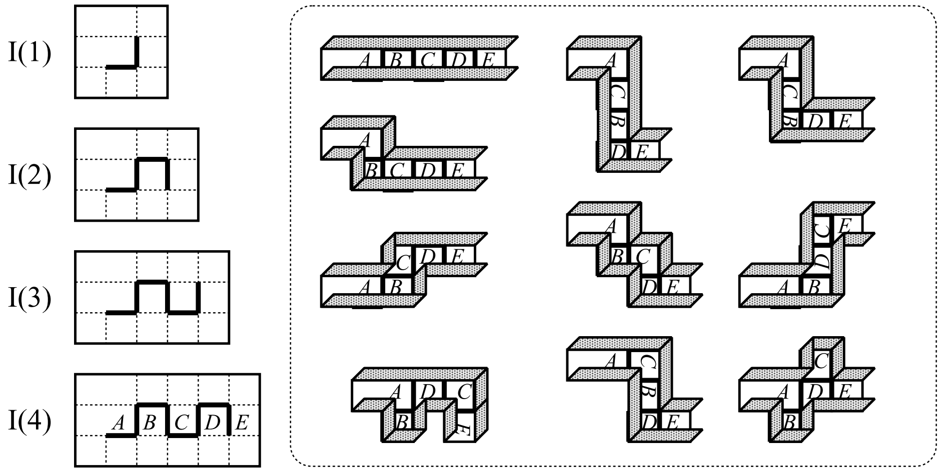

Fig. 6 I gadgets.

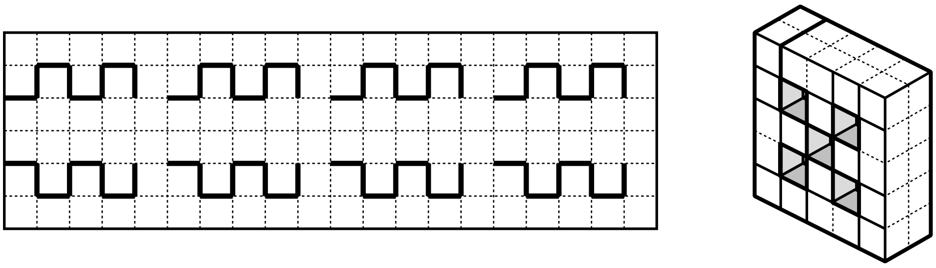

Fig. 7 A construction of a rectangle of size 6×20. It can fold to a polycube of genus 5.

fore, when a polycube Q is given, the algorithm can generate all unfoldings by generating all spanning trees for the graph $G(Q)$ (We note that, as mentioned in Introduction, some slits can be redundant in this context; however, we do not care about this issue. Therefore, some unfoldings in the figures contain redundant slits).

For each spanning tree of $G(Q)$, the algorithm checks whether the corresponding unfolding overlaps or not. If not, it gives a valid net of Q. If, moreover, it forms a rectangle, it is a solution of our problem. For a given spanning tree, this check can be done in linear time. Since all spanning trees of a given graph G can be enumerated in $O(1)$ time per tree (see Ref. [5]), all unfoldings of a given polycube Q of area n can be done in $O(nT(G(Q)))$ time, where $T(G(Q))$ is the number of spanning trees of $G(Q)$.

We note that our algorithm runs for any given polycube Q, and it can check if Q has a valid net or not. By exhaustive checking, we have the following theorem:

Theorem 4 All polycubes that consist of 12 or fewer cubes have an edge unfolding without overlapping.

We mention that Xu et al. investigated all polygons of area 30 that may fold to two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$ (and $\sqrt{5}\times \sqrt{5}\times \sqrt{5}$) by an exhaustive hybrid search of the breadth-first search and the depth-first search using a supercomputer [11]. Our exhaustive search also implies that it is quite hard to search the area of size much bigger than 30.

3. Proof of Theorem 2

Now we turn to construct a family of polycubes. We introduce a series of gadgets in **Fig. 6**. Let an *I gadget* of size i, denoted by $I(i)$, be a rectangle of size $3 \times (i + 2)$ with some slits given as shown in Fig. 6. That is, the $I(i)$ has a zig-zag slit of length $2i$ as shown in Fig. 6. This gadget can be folded not only in the I shape in a natural way, but also in many other ways. For example, $I(4)$ has ten ways of folding in total as shown in the right in Fig. 6. Therefore, in general, $I(i)$ has exponentially many ways of folding (The exact value is open, but it is at least $10^{i/4}$ by joining $i/4$ of $I(4)$s).

For the I gadget $I(4)$, we call the bottom left way of folding in Fig. 6 an *F-folding*. It is useful since it realizes a "turn" of direction. Gluing two copies of $I(4)$ gadgets (precisely, one is the mirror image) and performing F-folding, we can obtain an L-shaped pipe with two holes of size 1×2 on both endpoints. Therefore, joining four of the L-shaped pipes, we can construct a polycube as shown in **Fig. 7**. By elongating the gadgets, we can change the size and genus as shown in **Fig. 8**.

Combining these gadgets, it is easy to construct a rectangle with some slits for folding a polycube of any genus. In **Fig. 9** (a), we give an example of a rectangle with some slits that can be folded to a polycube of genus 2. Figure 9 (b) describes the polycube of genus 2 folded from (a) (since all polycubes folded in this manner are of thickness 2, we draw them in top-view).

We can observe that there are many polycubes folded from (a) by the property of the I gadget. That is, each I gadget in a rectangle can be folded to one of nine different shapes with consistency of lengths unless it intersects with others. That is, choosing each way of folding properly, we can fold to (exponentially) many different polycubes from the rectangle of length $6 \times n$ with slits. For a rectangle in Fig. 9 (a), one of the variants is given in Fig. 9 (c). Now it is easy to see that Theorem 2 and Corollary 3 hold.

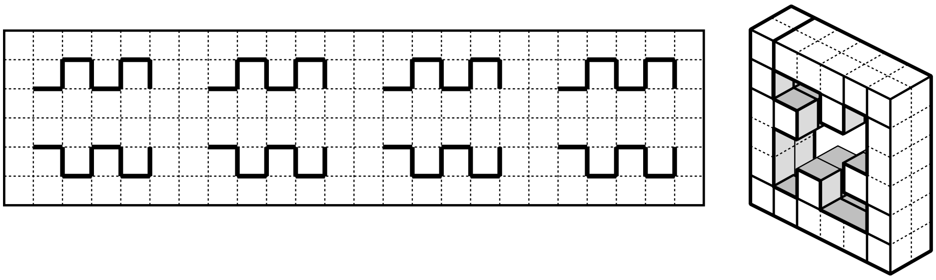

Fig. 8 A construction of rectangle of size 6×24. It can fold to a polycube of genus 1.

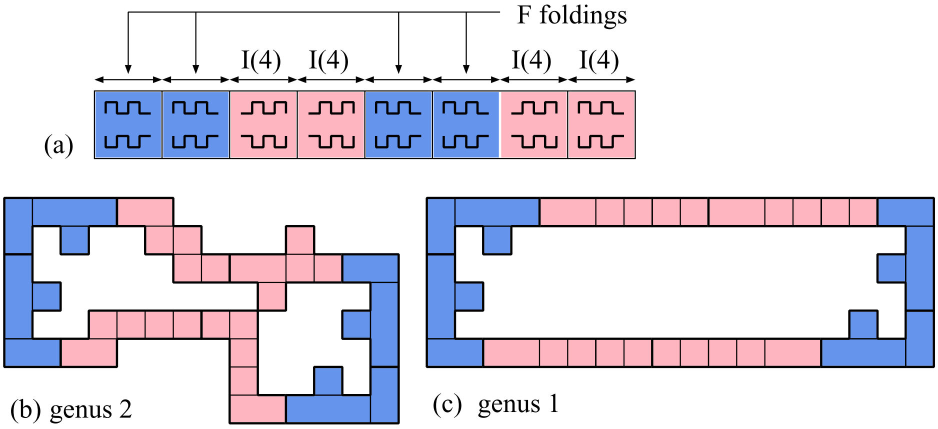

Fig. 9 (a) A construction of rectangle of size 6×48. (b) A polycube of genus 2 folded from the rectangle. (c) Another polycube of genus 1 folded from the same rectangle.

4. Concluding Remarks

In this paper, we show some concrete polycubes folded from a rectangle with slits. Among them, there is a polycube of genus 0. We also show that for any given integer k, one rectangle with slits can fold to at least k different polycubes. This construction also gives us at least k different polycubes of genus g for any positive integers k and g. So far, however, we have no construction that gives infinitely many polycubes of genus 0, which is an open problem.

The series of I gadgets in Fig. 6 gives us interesting patterns. For a given i, the number of ways of folding of I(i) seems to be an interesting problem from the viewpoint of computational origami. From the viewpoint of puzzle design, it is also an interesting problem to decide the kind of polyominoes folded from I(i) for general i. In the construction in Theorem 2 and Corollary 3, we use the rectangle of size $6 \times n$. It may be interesting whether we can use the rectangle of size $4 \times n$ or not.

In Theorem 4, we stated that all polycubes consisting of 12 or fewer cubes have an edge unfolding without overlaping. This theorem begins to address an open problem that asks whether there exists a polycube that has no non-overlapping edge unfolding. It seems very challenging to find such an "ununfoldable" polycube by brute-force search: our program is able to quickly find solutions for randomly sampled polycubes with as many as 1,000 cubes (as well as for hand-constructed polycubes that appear hard to unfold), and an exhaustive search becomes infeasible at much smaller numbers. This problem sometimes appears as "grid unfoldings" in the context of unfolding of orthogonal polyhedra. See Refs. [6], [7], [8], [9], [10] for further details.

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