

## Regular Paper

## Minimum Forcing Sets for Single-vertex Crease Pattern

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**Abstract:** We propose an algorithm for finding a minimum forcing set of a given flat-foldable *single-vertex crease pattern* (SVCP). SVCP consists of straight lines called *creases* that can be labeled as mountains or valleys, and the creases are incident to the center of a disk of paper. A *forcing set* is a subset of given creases that forces all other creases to fold according to the given labels. Our algorithm is a modification of an existing algorithm for 1D origami. We show that the size of a minimum forcing set of an SVCP is  $n/2$  or  $n/2 + 1$  where  $n$  is the number of creases in the SVCP.

**Keywords:** computational origami, single-vertex crease pattern, forcing set, flat-foldability

## 1. Introduction

In an origami application called self-folding origami, a thin material folds into an intended shape by rotating the planes around creases according to the label mountain or valley assigned on the creases (See Ref. [6], [8], [11], [12]). The cost of such an application can be reduced if it is enough to put actuators on a subset of creases. Finding such a subset of creases can be modeled as a forcing set problem. In applications, material is often desirable to satisfy *flat-foldability* in order to make the size small. A material is *flat-foldable* if we can transform it from the completely unfolded state to the flat state that all creases are completely folded.

The forcing set problem is a new topic in computational origami, which was considered in Ref. [1], [2], [4]. Especially, minimum forcing set for flat-foldability was studied for 1D origami [4] and 2D Miura-ori [2]. In a forcing set problem for flat-foldability, a flat-foldable *crease pattern*  $C$  and a flat-foldable *mountain-valley assignment* (or MV assignment briefly)  $\mu$  are given, where  $\mu$  is a function from creases to  $\{M, V\}$ . MV assignment  $\mu(c)$  on a crease  $c \in C$  determines the direction of rotation of the planes around  $c$  when folding. For a flat-foldable crease pattern  $C$  and its flat-foldable MV pattern  $\mu$ , a *forcing set*  $F$  is a subset of the creases in  $C$  such that if  $\mu'$  is any other flat-foldable MV assignment on  $C$  with  $\mu'(c) = \mu(c)$  for all  $c \in F$ , then we must have that  $\mu' = \mu$ . A forcing set  $F$  is called *minimum* if there is no other forcing set with size less than  $|F|$ .

This paper focuses on minimum forcing sets for flat-foldable *single-vertex crease pattern* (SVCP). An SVCP is a crease pattern whose creases are incident to the center of the sheet of paper to be folded. We consider the sheet of paper of SVCP is a disk. If  $|C|$  is two, we are to fold the disk in half, and it is obvious that

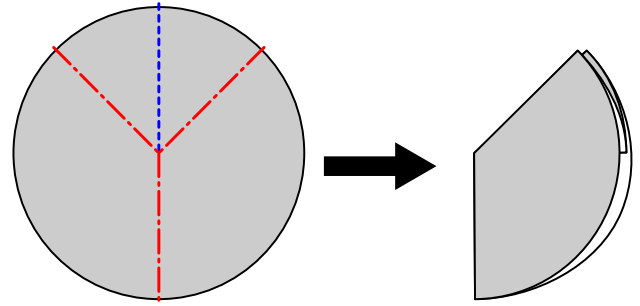


Fig. 1 An example of flat-foldable SVCP with MV assignment.

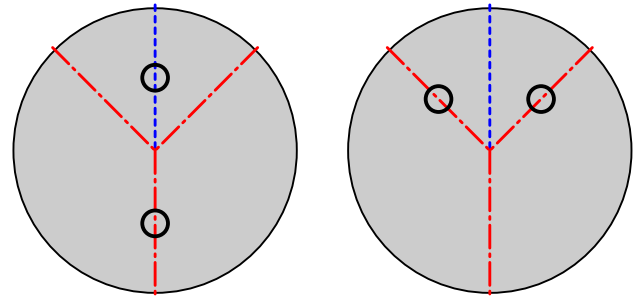


Fig. 2 Examples of minimum forcing sets for the flat-foldable SVCP in Fig. 1. Each crease with a small circle is the crease of the forcing set.

the size of the minimum forcing set is one. **Figure 1** is an example of flat-foldable SVCP with MV assignment. The minimum forcing sets for the flat-foldable SVCP in Fig. 1 are depicted in **Fig. 2**. A crease pattern is *flat-foldable* if and only if there exists an MV assignment so that the sheet of paper settles into a flat shape without penetrating itself after folding the creases along the assignment. Bern and Hayes developed an algorithm to determine flat-foldability of a given SVCP with MV assignment [3]. Flat-foldable SVCPs were studied in Ref. [13] from the viewpoint of enumeration as well. A crease pattern of SVCP is a sequence of creases  $C = (c_0, c_1, \dots, c_{n-1})$  which are put in clockwise on the disk incident to the center.  $\theta_i$  denotes the clockwise angle from  $c_i$  to  $c_{i+1 \bmod n}$ . We call  $(C, \mu)$  an *MV pattern*.

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In this paper, we develop an algorithm to find a minimum forcing set of a given flat-foldable SVCP in  $O(n^2)$  time. As far as the authors know, our algorithm is the first one for finding a minimum forcing set of flat-foldable SVCP, even though SVCP is an important component of origami. Our algorithm is based on one for 1D origami in Ref. [4] because the structure of SVCP is similar to 1D origami if we regard it as a ring by cutting away the inner space of the sheet of paper: the creases reduce to points on the ring, and the sheet of paper becomes 1D origami if we cut the ring at some point. We also reveal that the size of  $F$  is  $n/2$  or  $n/2 + 1$ . Precisely,  $|F|$  is  $n/2$  if the SVCP is of generic angles, which is a case that the angles to be operated always differ. In the case when all the angles in the SVCP are equal,  $|F|$  is  $n/2$  if  $n = 2$ , otherwise  $|F|$  is  $n/2 + 1$ . For a general SVCP, which does not have any constraints, the size of  $F$  is  $n/2 + 1$  if the crease pattern can be reduced to an SVCP of equal angles with size four or more by repeatedly crimping two consecutive creases  $(c_i, c_{i+1 \bmod n})$  with different MV assignment where  $\theta_i$  is minimal, otherwise  $|F| = n/2$ .

## 2. Preliminaries

This section introduces some terminology and preliminary results following [4]. Throughout the paper we work with a flat-foldable MV pattern  $(C, \mu)$ , where  $C = (c_0, c_1, \dots, c_{n-1})$  is an SVCP and  $\mu$  is a flat-foldable MV assignment.

### 2.1 Crimpable Sequences [4]

We slightly change the definition of crimpable sequence to fit the assumption that  $C$  is circular. A *crimpable* sequence in SVCP is composed of consecutive creases where the angles between the creases are equal, with the property that the two angles adjacent to the left and right end of the sequence are strictly larger than the equal angles. Formally, for integers  $0 \leq i < n$  and  $0 < k < n$ , a sequence of consecutive creases  $(c_i, c_{i+1 \bmod n}, \dots, c_{i+k \bmod n})$  is crimpable if  $\theta_i = \theta_{i+1 \bmod n} = \dots = \theta_{i+k-1 \bmod n}$  and  $\theta_{i-1 \bmod n} > \theta_i < \theta_{i+k \bmod n}$ . We note that we have to take a mod on the index for circulation. Thus we may have  $(i-1) \bmod n = (i+k) \bmod n$ .

A *monocrimp* operation is defined as a fold about a single pair of consecutive creases of opposite MV parity in a crimpable sequence.

A *crimp* operation is a set of monocrimps repeatedly conducted on a crimpable sequence while the sequence is crimpable. In our proofs for the minimum size of a forcing set, we characterize such size by considering the conditions for flat-foldability on a given SVCP while repeating a crimp operation to fold the SVCP flat. The following theorem will be needed in Section 5.

**Theorem 1 (Theorem 4 from Ref. [7])** Let  $\alpha$  be a crimpable sequence in a flat-foldable MV pattern. The difference in the number of M and V assignments for the creases in  $\alpha$  is zero (one) if  $\alpha$  has an even (odd) number of creases.

In the case of a crimpable sequence  $\alpha$  of odd length, we say that the crease remaining after a crimp operation on  $\alpha$  *survives* the crimp. We note that the surviving crease in  $\alpha$  is with *majority assignment* in  $\alpha$  (Ref. [4], Observation 1). Majority assignment denotes the assignment M or V which is major in a sequence or a set of creases.

### 2.2 End Creases [4]

*End creases* are the remains after *exhaustive crimps*. Exhaustive crimps mean crimping repeatedly until there is no crimpable sequence. The following lemma holds for SVCP.

**Lemma 2** The end creases of an SVCP form a flat-foldable SVCP of equal angles.

To prove this lemma, we need the following lemma and theorem:

**Lemma 3 (Corollary 12.2.11 from Ref. [5])** An equal-angle SVCP is flat-foldable if and only if  $|\#M - \#V| = 2$ .

**Theorem 4 (The Maekawa Theorem)** In a flat-foldable SVCP with MV assignment defined by angles  $\theta_0 + \theta_1 + \dots + \theta_{n-1} = 360^\circ$ , the number of mountains and the number of valleys differ by  $\pm 2$ .

Details about the Maekawa Theorem can be found in Ref. [5], Chapter 12. Now let us prove Lemma 2.

**Proof.** We will make exhaustive crimps, that is, we will repeat crimps while processed  $C$  satisfies  $\theta_{i-1 \bmod n} > \theta_i = \theta_{i+1 \bmod n} = \dots = \theta_{i+k-1 \bmod n} < \theta_{i+k \bmod n}$  for some  $i$  and  $k$ . While two or more different values of angles exist in  $C$ , this condition is always satisfied around the minimum angle. Thus, after this repetition, the crease pattern becomes one that consists of all equal angles.

The original flat-foldable  $(C, \mu)$  satisfies the equation in Lemma 3 by the Maekawa Theorem. A monocrimp does not change the difference between the number of Ms and the number of Vs. Therefore after crimping all crimpable sequences in  $(C, \mu)$ , the crease pattern satisfies  $|\#M - \#V| = 2$ . By Lemma 3, the obtained equal-angle SVCP is flat-foldable. ■

## 3. The Size of a Minimum Forcing Set of an SVCP

This section is devoted to proof of the theoretical minimum size of forcing sets.

### 3.1 SVCP of Generic Angles

In this section, let a given SVCP be of generic angles, that is, consecutive angles to be crimped always differ. Formally, SVCP is of *generic angles* if  $\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3} + \dots + \theta_{j-1} \neq \theta_j - \theta_{j+1} + \theta_{j+2} - \theta_{j+3} + \dots + \theta_{k-1}$  holds for any  $i, j$ , and  $k$  such that  $i \leq j \leq k \pmod{n}$  where the length of each sequence is odd except when the inequality includes all of the angles  $\theta_i$  around the SVCP. (This definition is from Ref. [5], Section 12.2.2. We here note that when SVCP is flat-foldable, the last two creases should be separated by equal angles.) The following lemma is important in our proof.

**Lemma 5 (from Refs. [9], [10])** If an angle  $\theta_i$  is strictly minimal, that is,  $\theta_{i-1 \bmod n} > \theta_i < \theta_{i+1 \bmod n}$  holds, then the two consecutive creases  $c_i$  and  $c_{i+1 \bmod n}$  forming  $\theta_i$  have assignment different from each other in any flat-foldable MV pattern.

In this section, first we show the existence of  $F$  with size  $n/2$ , then we prove that  $F$  with size  $n/2 - 1$  or less does not exist.

**Lemma 6** There is a forcing set of an SVCP of generic angles, whose size is  $n/2$ .

**Proof.** First we use a contradiction in order to show that there are always three consecutive angles which satisfy Lemma 5. Assume there are different consecutive angles  $\theta_0, \theta_1, \dots, \theta_{n-1}$ , and any consecutive three of them do not satisfy Lemma 5. Then,

for example, we can assume  $\theta_0 > \theta_1 > \theta_2$ . By the condition and assumption,  $\theta_1 > \theta_2 > \theta_3$  holds. Similarly,  $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4 > \dots$  holds and the sequence monotonically decreases. However,  $\theta_{n-1} > \theta_0$  could never happen, which is a contradiction. Therefore we can fold the SVCP by repeatedly finding three consecutive angles  $\theta_{i-1 \bmod n}, \theta_i, \theta_{i+1 \bmod n}$  satisfying Lemma 5 and applying the monocrimp operation.

Applying Lemma 5 on  $(\theta_{i-1 \bmod n}, \theta_i, \theta_{i+1 \bmod n})$  repeatedly, we can fold flat the sheet of paper. Let  $(c_i, c_{i+1 \bmod n})$  be the pair of creases between the three angles. If we determine the assignment on one of  $(c_i, c_{i+1 \bmod n})$ , the assignment on the other of the pair is also determined. Hence we can make a forcing set by picking a crease in each pair as an element of the forcing set. Therefore the size of the forcing set is  $n/2$ . ■

**Lemma 7** There is no forcing set of an SVCP of generic angles whose size is less than  $n/2$ .

**Proof.** The proof is by contradiction. Assume a forcing set  $F$  with size  $n/2 - 1$  or less exists.

We monocrimp  $(\theta_{i-1 \bmod n}, \theta_i, \theta_{i+1 \bmod n})$  according to Lemma 5 except the last two creases. Every pair  $(c_i, c_{i+1 \bmod n})$  is isolated from other pairs and there are  $n/2$  pairs (including the last pair), thus every crease appears in a pair only once. Because  $|F| < n/2$ , there is an index  $i$  such that both in  $(c_i, c_{i+1 \bmod n})$  are not in  $F$ . This contradicts the definition of  $F$  because the sheet of paper folds flat in the following two cases when it is not the last pair: we assign  $(M, V)$  on  $(c_i, c_{i+1 \bmod n})$ , or  $(V, M)$  on  $(c_i, c_{i+1 \bmod n})$ . When it is the last pair, we assign  $(M, M)$  or  $(V, V)$  on it, which contradicts the definition of  $F$  again. Thus there is no forcing set of size less than  $n/2$ . ■

By Lemma 6 and Lemma 7, we obtain the following theorem.

**Theorem 8** The size of a minimum forcing set for SVCP of generic angles is  $n/2$ .

### 3.2 SVCP of Equal Angles

In this section, let a given SVCP be of equal angles, or *equal-angle* SVCP. Hence  $\theta_i = \theta_{i+1 \bmod n}$  holds for any integer  $i$ .

**Lemma 9** There is a forcing set of an equal-angle SVCP whose size is  $n/2 + 1$  if  $n \geq 4$ . Furthermore, the forcing set is composed of all creases with majority assignment.

**Proof.** Assume that  $F$  consists of all majority M creases (thus all V creases are not in  $F$ ). If  $F$  is not a forcing set then we can choose some crease in  $C \setminus F$  to be M, contradicting Lemma 3. ■

**Lemma 10** There is no forcing set of an equal-angle SVCP whose size is less than  $n/2 + 1$  if  $n \geq 4$ .

**Proof.** We prove it by contradiction. Assume  $F$  is a forcing set of an equal-angle SVCP, whose size is  $n/2$  or less. Then there may be a pair of an M crease and a V crease which are not in  $F$  (let M be the majority in the MV pattern). We denote such pair by  $p$ . We note that the creases in  $p$  do not have to be consecutive.

If all V creases are in  $F$ ,  $p$  does not exist. In this case, we can invert the assignment of a pair of M creases in  $C \setminus F$  to Vs, where the pair is not necessary to be consecutive. This operation contradicts Lemma 3.

Otherwise we can swap the MV assignment in  $p$ , and the resulting SVCP is flat-foldable by Lemma 3. This is a contradiction to our assumption that  $F$  is forcing. ■

**Theorem 11** Assume that a given SVCP is of equal angles. If the number of creases in the SVCP is two, then the minimum forcing set consists of one crease. Otherwise the size of the minimum forcing set of the SVCP is  $n/2 + 1$ . By Iverson's convention, it can be described as  $n/2 + [n \geq 4]$ .

**Proof.** It is obvious if the number of creases in an equal-angle SVCP is two. Lemma 9 and Lemma 10 imply that  $n/2 + 1$  is the minimum size of  $F$  if  $n \geq 4$ . ■

### 3.3 General SVCP

Here we consider that a given SVCP has no constraints.

**Theorem 12** Let  $m$  be the number of monocrimps performed until the given SVCP becomes a flat-foldable equal-angle SVCP (cf. Lemma 2).  $F$  denotes a minimum forcing set of the given SVCP. Then  $|F| = n/2 + [n - 2m \geq 4]$ .

**Proof.** As the case of generic angles, we crimp the creases in crimpable sequences as possible as we can. For each monocrimp, one of the creases in the pair must be in  $F$ . Such monocrimps contribute to  $m$  elements in  $F$ .

After monocrimping  $m$  times, the crease pattern has become a flat-foldable equal-angle SVCP (cf. Lemma 2). This equal-angle SVCP is composed of  $n - 2m$  creases because two creases are consumed per one monocrimp. By Theorem 11, the size of a minimum forcing set of the equal-angle SVCP is  $(n - 2m)/2 + [n - 2m \geq 4]$ .

The minimum size of  $F$  is the sum of the sizes of the two sets of forcing creases obtained above. This is because the sets do not have an intersection and both are minimum. Thus,  $|F| = m + (n - 2m)/2 + [n - 2m \geq 4] = n/2 + [n - 2m \geq 4]$ . ■

## 4. Constructing a Minimum Forcing Set

### 4.1 Crimp Forest Construction

We convert the crimp forest algorithm in Ref. [4] to an algorithm for SVCP by allowing circulation of the index of creases when finding a crimpable sequence. The converted algorithm is shown in Algorithm 1. A circulating crimpable sequence  $(c_i, c_{i+1}, \dots, c_0, c_1, \dots, c_k)$  may occur when the algorithm finds and crimps crimpable sequences, but it does not change the behavior of the other parts of the algorithm. The algorithm constructs a forest in a bottom-up manner. The edges of the forest are added if the sequence in the parent node includes the crease surviving the crimp on the sequence in a child node. **Figure 3** depicts an example of a crimp forest.

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#### Algorithm 1: CRIMPFORSTSVCP( $C, \mu$ )

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Initialize  $W \leftarrow \emptyset$ ;

**while**  $C$  has a crimpable sequence **do**

  let  $s$  be the crimpable sequence in  $C$  with the smallest starting index; // modified from Ref. [4]

  create a node  $v$  corresponding to  $s$ , and add  $v$  to  $W$ ;

  make  $v$  the parent of each root node in  $W$  whose crimpable

  sequence has a surviving crease that is in  $s$ ;

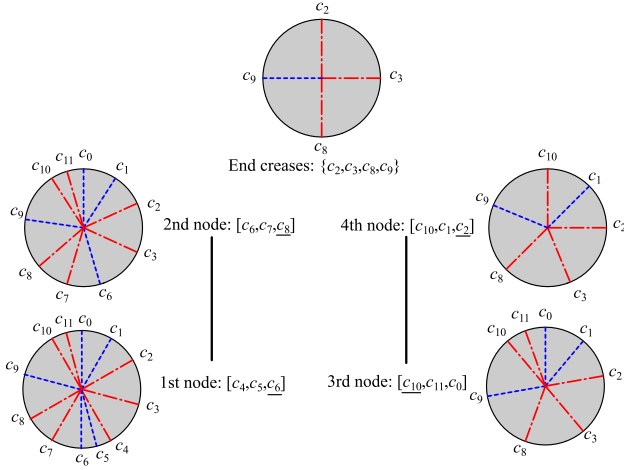
  apply the crimp operation to  $s$ ;

  update  $C$  to be the resulting crease pattern;

**end**

**return**  $W$ .

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**Fig. 3** An example of a crimp forest construction.  $[-]$  is a crimpable sequence to be crimped. The surviving creases are underlined. Inclusion of a surviving crease is presented as an edge of a tree. We note that each “disk” actually forms a “cone”. It depicts the top view of the cone.

A straightforward implementation of Algorithm 1 takes  $O(n^2)$  time because a naive way to find a crimpable sequence takes  $O(n)$  time: start searching from  $c_0$  in clockwise; skip monotonically nonincreasing angles; stop at the right side crease  $c_r$  which satisfies  $\theta_{r-1 \bmod n} < \theta_r$ ; in counterclockwise from  $c_r$ , search the left side crease  $c_l$  which satisfies  $\theta_{l-1 \bmod n} > \theta_l$ ; other operations can be done in constant time; since the algorithm loops at most  $n$  times, the time complexity of the algorithm is  $O(n^2)$ .

The following lemma describing the properties of crimp forest holds for SVCP as well.

**Lemma 13 (Lemma 4 from Ref. [4])** Given a crease pattern  $C$  and two flat-foldable MV assignments  $\mu_1$  and  $\mu_2$ , let  $W_1$  and  $W_2$  be the crimp forests corresponding to  $(C, \mu_1)$  and  $(C, \mu_2)$ , respectively. Then the following properties hold:

- (1)  $W_1$  and  $W_2$  are structurally identical.
- (2) Corresponding nodes in  $W_1$  and  $W_2$  have crimpable sequences of the same size and the same interval distances between adjacent creases.
- (3) Creases involved for the first time in a crimpable sequence at a node in  $W_1$  have the same position in the crimpable sequence at the corresponding node in  $W_2$ .

#### 4.2 Forcing Set Algorithm

We convert the forcing set algorithm in Ref. [4]. We switch  $\text{CRIMPFOREST}(C, \mu)$  in Ref. [4] to  $\text{CRIMPFORESTSVCP}(C, \mu)$ ; it first simulates  $\text{CRIMPFOREST}(C, \mu)$  with initialization of  $F$  to all end creases as in  $\text{CRIMPFOREST}(C, \mu)$ , and later it removes redundant creases in the end creases. See Algorithm 2 for the detail.

We note that Algorithm 3 is the same as the corresponding algorithm in Ref. [4]. Only Algorithm 2 differs.

The preorder traversal takes  $O(n)$  time because each node is visited only once and the sum of lengths of the sequences in the nodes is  $n$ . Thus the main factor of computation time is  $\text{CRIMPFORESTSVCP}$ , which takes  $O(n^2)$  time.

We need the following lemma for the proof in Section 5:

**Lemma 14 (Lemma 6 from Ref. [4])** Let  $(C, \mu_1)$  be a flat-foldable MV pattern, and let  $F$  be the forcing set generated by

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#### Algorithm 2: $\text{FORCINGSETSVCP}(C, \mu)$

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Initialize  $W$  to the output generated by  $\text{CRIMPFORESTSVCP}(C, \mu)$ ;
//  $W$  is the crimp forest
initialize  $EC$  to the set of end creases that remain after running
 $\text{CRIMPFORESTSVCP}(C, \mu)$ ;
//  $EC$  forms a cyclic creases with equal angles
initialize  $F$  to the creases in  $EC$ ;
 $\text{FORCINGSET1D}(C, W, F, \mu)$ ;
if  $|EC| = 2$  then remove one crease from  $F \cap EC$ ;
if  $|EC| \geq 4$  then remove the creases with minority assignment in  $EC$ 
from  $F \cap EC$ ;
output  $F$ .

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#### Algorithm 3: $\text{FORCINGSET1D}(C, W, F, \mu)$

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foreach tree  $T \in W$  do
  foreach node  $v$  in a preorder traversal of  $T$  do
    if  $v$ 's crimpable sequence has even length;
    then
      add to  $F$  all creases from  $v$ 's crimpable sequence having
      M assignment;
    else if the surviving crease from  $v$ 's crimpable sequence is
    already in  $F$  then
      add to  $F$  all creases from  $v$ 's crimpable sequence having
      the majority MV assignment;
    else
      add to  $F$  all creases from  $v$ 's crimpable sequence having
      the minority MV assignment.
    end
  end
end

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Algorithm 2 with input  $(C, \mu_1)$ . Let  $(C, \mu_2)$  be a flat-foldable pattern such that  $\mu_2$  agrees with  $\mu_1$  on the forcing set  $F$ , that is,  $\mu_2(c) = \mu_1(c)$  for  $c \in F$ . Let  $T_1$  and  $T_2$  be two structurally equivalent trees generated by the forcing set algorithm with inputs  $(C, \mu_1)$  and  $(C, \mu_2)$ , respectively. If a crease  $c$  in a crimpable sequence  $\alpha_1 \in T_1$  is in  $F$ , then a crease (not necessarily  $c$ ) with the same MV assignment occurs in the corresponding crimpable sequence  $\alpha_2 \in T_2$ , in the same position as in  $\alpha_1$ .

### 5. Proof of Correctness

For a given SVCP, a forcing set is of size  $n/2$  or  $n/2 + 1$  if it is flat-foldable. This section proves that  $F$  created by Algorithm 2 is forcing and minimum. The proof is almost the same as Ref. [4] because Damian et al. use local properties of crimpable sequence and abstract properties of crimp forest, which are not affected by the change from 1D to SVCP except the last step, where we have an SVCP of equal angles eventually.

We consider the set  $F_0$  after  $\text{FORCINGSET1D}(C, W, F, \mu)$ . We first observe that the algorithm is the same as the one in Ref. [4] up to this step. Therefore,  $F_0$  is a forcing set. Let  $F_1$  be the set after the removal of creases from  $F_0 \cap EC$ . Then it is sufficient to show that  $F_1$  is still a forcing set after removal, and it has the minimum size stated in Theorem 12.

We here observe that, in Algorithm 3, each node  $v$  in a tree  $T$  in the crimp forest  $W$  provides (1)  $n_v/2$  creases to  $F$  for even  $n_v$ , (2a)  $(n_v - 1)/2$  creases to  $F$  for odd  $n_v$  if  $v$  has no surviving crease in  $F$ , and (2b)  $(n_v + 1)/2$  creases to  $F$  for odd  $n_v$  if  $v$  has a surviving crease in  $F$ , where  $n_v$  is the number of creases in  $v$ . We note



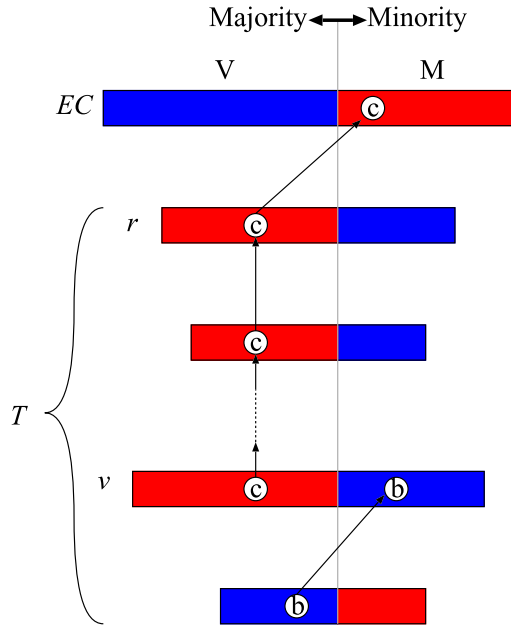


Fig. 4 A crease  $c$  which is removed from  $F \cap EC$  after computing  $F$ .

that, for a node  $v$  of odd size  $n_v$ , the surviving crease in majority assignment will be counted in its parent node. Therefore, it is not difficult to see that  $F_1$  has the same size of the minimum forcing set stated in Theorem 12.

We here prove that  $F_1$  is a forcing set. To derive contradictions, we assume that  $F_1$  is not a forcing set. In the set  $EC$  of end creases, since the set forms an SVCP of equal angles, we can observe that the set  $EC \setminus (F_0 \setminus F_1)$  is a forcing set with respect to  $EC$ . Thus, there is a crease  $c$  in  $F_0 \setminus F_1$  that violates the property of a forcing set with respect to a tree  $T$  in the crimp forest  $W$ . Without loss of generality, we assume that the minority assignment in  $EC$  is  $M$ . Let  $r$  be the root of  $T$ . Then the nodes including  $c$  induces a path from  $r$  to a node in  $T$  (Fig. 4). Let  $v$  be the lowest node that includes  $c$ . Then, every node from  $r$  to  $v$  has a majority assignment  $M$ , and  $c$  is the surviving crease from  $v$  to  $EC$ . Along the path from  $r$  to  $v$ , all creases with a majority assignment  $M$  are put in  $F$  by Algorithm 3. Since  $c$  violates the property of the forcing set, we have a node  $u$  along the path from  $r$  to  $v$  such that  $F_1$  is not a forcing set any more. Now the node  $u$  has  $n_u$  creases, the number  $n_u$  is odd, and all creases with assignment  $M$  but  $c$  is in  $F_1$ . That is,  $(n_u - 1)/2$  creases with assignment  $M$  are in  $F_1$ , the crease  $c$  is not in  $F_1$ , and the other  $(n_u - 1)/2$  creases have assignment  $V$ . Since  $F_1$  is not a forcing set, we have a crease  $c'$  with assignment  $V$  such that the assignments  $V$  of  $c'$  and  $M$  of  $c$  can be exchanged without breaking the flat-foldability. We here note that any crease with assignment  $V$  can be chosen, however, we have to choose  $c$  with assignment  $M$  because this is the only crease that is not in  $F_1$ . However, if we change the assignment of  $c$  from  $M$  to  $V$ , it contradicts that the set  $EC$  of end creases is flat-foldable. Therefore, any crease in  $EC$  with minority assignment does not violate the property of a forcing set even if it is removed from  $F_0$ .

Therefore, the output  $F_0$  is the minimum forcing set, which completes the proof of the correctness of the algorithm.

## 6. Conclusion

We have developed an algorithm to find a minimum forcing set of flat-foldable SVCP in  $O(n^2)$  time. We have shown that the size of such forcing set is  $n/2$  or  $n/2 + 1$ . It is an open problem to find a minimum forcing set of arbitrary 2D origami. Considering the case of two vertices might be the first step to solve the arbitrary case. Enumeration of minimum forcing sets of a given MV pattern is an interesting problem as well. We believe that our result will help us to solve such open problems.

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## References

- [1] Abel, Z., Cantarella, J., Demaine, E.D., Eppstein, D., Hull, T.C., Ku, J.S., Lang, R.J. and Tachi, T.: Rigid origami vertices: Conditions and forcing sets, *Computational Geometry*, Vol.7, No.1, pp.171–184 (2016).
- [2] Ballinger, B., Damian, M., Eppstein, D., Flatland, R., Ginepro, J. and Hull, T.: Minimum forcing sets for Miura folding patterns, *Proc. 26th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.136–147, Society for Industrial and Applied Mathematics (2015).
- [3] Bern, M. and Hayes, B.: The complexity of flat origami, *Proc. 7th Ann. ACM-SIAM Symp. Discrete Algorithms*, pp.175–183 (1996).
- [4] Damian, M., Demaine, E., Dulieu, M., Flatland, R., Hoffman, H., Hull, T.C., Lynch, J. and Ramaswami, S.: Minimum forcing sets for 1D origami, arXiv preprint arXiv:1703.06373v1 (2015).
- [5] Demaine, E.D. and O’Rourke, J.: *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*, Cambridge University Press (2007).
- [6] Hawkes, E., An, B., Benbernou, N.M., Tanaka, H., Kim, S., Demaine, E.D., Rus, D. and Wood, R.J.: Programmable matter by folding, *Proc. National Academy of Sciences*, Vol.107, No.28, pp.12441–12445 (online), DOI: 10.1073/pnas.0914069107 (2010).
- [7] Hull, T.: Counting mountain-valley assignments for flat folds, *Ars Combinatoria*, Vol.67, pp.175–187 (2003).
- [8] Ionov, L.: 3D microfabrication using stimuli-responsive self-folding polymer films, *Polymer Reviews*, Vol.53, No.1, pp.92–107 (2013).
- [9] Justin, J.: Towards a mathematical theory of origami, *Proc. 2nd International Meeting Origami Science, Technology*, pp.15–29 (1994).
- [10] Kawasaki, T.: On the relation between mountain-creases and valley-creases of a flat origami, *Proc. 1st International Meeting Origami Science, Technology*, pp.229–237 (1989).
- [11] Leong, T.G., Lester, P.A., Koh, T.L., Call, E.K. and Gracias, D.H.: Surface Tension-Driven Self-Folding Polyhedra, *Langmuir*, Vol.23, No.17, pp.8747–8751 (online), DOI: 10.1021/la700913m (2007).
- [12] Mahadevan, L. and Rica, S.: Self-Organized Origami, *Science*, Vol.307, No.5716, pp.1740–1740 (online), DOI: 10.1126/science.1105169 (2005).
- [13] Ouchi, K. and Uehara, R.: Efficient enumeration of flat-foldable single vertex crease patterns, *IEICE Trans. Information and Systems*, Vol.E102-D, No.3, pp.416–422 (2019).



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