Minimizing a Vertex Set Satisfying Specific Graph Properties

Yuma Tamura[1,a)] Takehiro Ito[1,b)] Xiao Zhou[1,c)]

Abstract: Let $\Pi_1, \Pi_2, \ldots, \Pi_c$ be graph properties for a fixed integer c. Then, $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is the problem of asking whether the vertex set of a given graph can be partitioned into c subsets $V_1, V_2, \ldots, V_c$ such that the subgraph induced by V_i satisfies the graph property Π_i for every $i \in \{1, 2, \ldots, c\}$. Minimization and parameterized variants of $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition have been studied for several specific graph properties, where the size of the vertex subset V_1 satisfying Π_1 is minimized or taken as a parameter. In this paper, we first show that the minimization variant is hard to approximate for any nontrivial additive hereditary graph properties, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. We then give FPT algorithms for the parameterized variant when restricted to the case where $c = 2$, Π_1 is a hereditary graph property, and Π_2 is the class of acyclic graphs.

Keywords: Graph Algorithm, Inapproximability, Independent Feedback Vertex Set, Bipartite Graph

1. Introduction

Various combinatorial problems on graphs can be seen as problems of partitioning the vertex set of a given graph into a fixed number of vertex subsets satisfying prescribed properties. For example, c-Coloring is the problem of deciding whether the vertex set of a given graph can be partitioned into c independent sets (i.e., edgeless graphs). Another example is Near-Bipartiteness, which is the problem of deciding whether the vertex set of a given graph can be partitioned into two subsets such that one forms an independent set and the other forms an acyclic graph. These problems can be unified as the problem $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition for a fixed integer c, where $\Pi_1, \Pi_2, \ldots, \Pi_c$ denote graph properties: $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition, also known as Generalized Graph Coloring [1], is the problem of asking whether the vertex set of a given graph can be partitioned into c subsets $V_1, V_2, \ldots, V_c$ such that the subgraph induced by V_i satisfies the graph property Π_i for every $i \in \{1, 2, \ldots, c\}$. We call such a vertex partition a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$*-coloring* of the graph. (See Fig. 1 as an example.) Minimization and parameterized variants of $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition have been also studied in the literature for several graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, where the size of the vertex subset V_1 satisfying Π_1 is minimized or taken as a parameter.

We here define some terms for graph properties. A *graph property*, or simply a *property*, is a property of graphs closed under isomorphism. We sometimes regard a graph property as a class of graphs (i.e., a set of all graphs) satisfying the property. A graph property Π is *hereditary* if, for any graph G satisfying Π, every induced subgraph of G also satisfies Π. A graph property Π is *additive* if, for any two graphs G and H satisfying Π, the disjoint union of G and H also satisfies Π, where the *disjoint union* of $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph whose vertex set is $V_G \cup V_H$ and edge set is $E_G \cup E_H$. A graph property Π is *nontrivial* if there exists at least one graph satisfying Π and there exists at least one graph which does not satisfy Π.

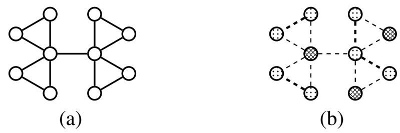

Fig. 1 (a) A graph G, and (b) a (Π_1, Π_2)-coloring of G, where Π_1 is the class of edgeless graphs and Π_2 is the class of path graphs. The number of hatched vertices is minimum among all (Π_1, Π_2)-colorings of G.

1.1 Related Results and Known Results

Farrugia [3] showed that $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is NP-hard for any fixed nontrivial additive hereditary graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. Notice that if $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs, then the problem is equivalent to 2-Coloring and hence it can be solved in linear time for general graphs.

Kanj et al. [7] widely studied the parameterized complexity of (Π_1, Π_2)-Partition. They mentioned that a simple branching technique yields a single-exponential FPT algorithm for Parameterized (Π_1, Π_2)-Partition if Π_1 and Π_2 are hereditary graph properties such that the membership of Π_1 can be decided in polynomial time and Π_2 can be characterized by a finite set of forbidden induced subgraphs.

Many FPT algorithms have been developed for various prob-

1 Graduate School of Information Sciences, Tohoku University, Aobayama 6-6-05, Sendai, 980-8579, Japan
a) yuma.tamura.t5@dc.tohoku.ac.jp
b) takehiro@tohoku.ac.jp
c) zhou@tohoku.ac.jp

lems, which can be seen as Parameterized (Π_1, Π_2)-Partition with specific graph properties Π_1 and Π_2, such as Feedback Vertex Set [6], Independent Feedback Vertex Set [8], [12], and $\mathcal{G}$-Bipartization [11]. On the other hand, Parameterized (Π_1, Π_2)-Partition is fixed-parameter intractable even if Π_1 is the class of all graphs: the problem is $W[P]$-complete if Π_2 is the class of d-degenerate graphs for any $d \geq 2$ (this corresponds to d-Degenerate Vertex Deletion) [10], and the problem is $W[2]$-hard if Π_2 is the class of wheel-free graphs (this corresponds to Wheel-Free Deletion) [9].

From the viewpoint of approximation, there is a polynomial-time 2-approximation algorithm for Feedback Vertex Set [2], which is equivalent to Min (Π_1, Π_2)-Partition if Π_1 is the class of all graphs and Π_2 is the class of acyclic graphs. However, if we change Π_1 to the class of edgeless graphs, then the problem is equivalent to Independent Feedback Vertex Set and it is hard to approximate even for planar bipartite graphs [14].

1.2 Our Contribution

In this paper, we study the approximability of Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition and the fixed-parameter tractability of Parameterized (Π_1, Π_2)-Partition.

We first study the approximability. It is already NP-hard to decide if a given graph has at least one $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring for nontrivial additive hereditary graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$ [3]. In this paper, we give inapproximability results of Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition even for the case where we know that a given graph has at least one $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring. We show that Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition, any fixed $c \geq 2$, is hard to approximate for any fixed nontrivial additive hereditary graph properties, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. In addition, we show that Min (Π_1, Π_2)-Partition for planar bipartite graphs remains hard to approximate if each of Π_1 and Π_2 has a minimal forbidden induced subgraph that is planar and bipartite. Interestingly, as we will discuss in Section 3, Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition can be solved in polynomial time for bipartite graphs if $c \geq 3$ and $\Pi_1, \Pi_2, \ldots, \Pi_c$ are nontrivial additive hereditary graph properties. We note that various well-known graph properties are additive and hereditary: for example, the classes of acyclic graphs, interval graphs, planar graphs, and more generally, $\mathcal{H}$-free graphs for a graph family $\mathcal{H}$.

We then investigate the fixed-parameter tractability of Parameterized $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition when restricted to $c = 2$ and Π_2 is the class of acyclic graphs. We first develop an FPT algorithm for the problem if Π_1 is a hereditary graph property; we also show that the running time can be improved for bounded degeneracy graphs. Note that this result cannot be covered by [7], because the class of acyclic graphs is characterized by the infinite forbidden cycles. We then give an FPT algorithm for the case where Π_1 is the class of graphs with maximum degree Δ, for a fixed Δ. We also develop a faster FPT algorithm when restricted to $\Delta = 1$.

2. Preliminaries

In this paper, we assume that graphs are simple, finite, undirected, and unweighted. Let $G = (V, E)$ be a graph. We sometimes denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For a vertex subset V' of G, let $G[V']$ be the subgraph of G induced by V'. We denote simply by $G - V'$ the induced subgraph $G[V \setminus V']$. We say that an induced subgraph H of G is *proper* if $V(G) \setminus V(H) \neq \emptyset$. For a vertex v in G and a vertex subset $V' \subseteq V$, we denote by $N(v, V')$ the set of all neighbors of v in $G[V' \cup \{v\}]$, that is, $N(v, V') = \{w \in V' : vw \in E\}$. We denote by $d(v, V')$ the degree of v in $G[V' \cup \{v\}]$, that is, $d(v, V') = |N(v, V')|$.

We have already defined the terms *graph property*, *hereditary*, *additive*, and *nontrivial* in Introduction. Recall that we sometimes regard a graph property as a class of graphs (i.e., a set of all graphs) satisfying the property. For a property Π, a graph is said to be a *forbidden induced subgraph* for Π if it does not satisfy Π. A forbidden induced subgraph H is said to be *minimal* if any proper induced subgraph of H satisfies Π. A *minimal forbidden set* $\mathcal{F}(\Pi)$ of Π is a set of all minimal forbidden induced subgraphs for Π. Any additive hereditary property can be characterized by a (possibly infinite) minimal forbidden set $\mathcal{F}(\Pi)$ such that every graph in $\mathcal{F}(\Pi)$ is connected. Moreover, if the property is nontrivial, every graph in $\mathcal{F}(\Pi)$ has at least two vertices. For example, $\mathcal{F}(\Pi) = \{K_2\}$ if Π is the class of edgeless graphs, and $\mathcal{F}(\Pi') = \{C_3, C_4, C_5, \ldots\}$ if Π' is the class of acyclic graphs, where K_n is a complete graph of n vertices and C_n is a cycle of n vertices.

In the remainder of this paper, we regard a partition of the vertex set of a graph G as a (vertex) coloring of G. Let $C = \{1, 2, \ldots, c\}$ be a color set, where c is a positive integer. Then, a *coloring* of G is simply a mapping $f : V(G) \to C$. For properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, a coloring f of G is called a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-*coloring* of G if $G[f^{-1}(i)]$ satisfies Π_i for every $i \in C$. We say that a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring f of G is *optimal* if $|f^{-1}(1)|$ is minimum among all $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-colorings of G. We define $\mathsf{OPT}(G)$ as follows:

$$\mathsf{OPT}(G) = \min\{|f^{-1}(1)| : f \text{ is a } (\Pi_1, \Pi_2, \ldots, \Pi_c)\text{-coloring of } G\}$$

if G has a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring; otherwise we let $\mathsf{OPT}(G) = +\infty$. For fixed properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, we define Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition as the problem of computing $\mathsf{OPT}(G)$ for a given graph G. We also study the problem parameterized by the solution size k: Parameterized $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is the problem of determining whether $\mathsf{OPT}(G) \leq k$ or not.

3. Inapproximability

In this section, we study the inapproximability of Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition. We say that an algorithm for Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is $\rho(n)$-*approximation* if it returns a value z for a given graph G with n vertices such that $z \leq \rho(n) \cdot \mathsf{OPT}(G)$ and G has a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring f satisfying $|f^{-1}(1)| = z$. Then, $\mathsf{OPT}(G) \leq z \leq \rho(n) \cdot \mathsf{OPT}(G)$ always holds, and hence the algorithm must compute $\mathsf{OPT}(G)$ if either $\mathsf{OPT}(G) = 0$ or $\mathsf{OPT}(G) = +\infty$ holds. In this section, we give inapproximability results that hold even if we know that a given graph G satisfies both $\mathsf{OPT}(G) \neq 0$ and $\mathsf{OPT}(G) \neq +\infty$. We say that a graph G is *promised* if both $\mathsf{OPT}(G) \neq 0$ and $\mathsf{OPT}(G) \neq +\infty$ hold.

The main result of this subsection is the following theorem.

Theorem 1. *Let Π_1 and Π_2 be any two fixed nontrivial additive hereditary graph properties. Let G be a promised graph of n vertices, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Under the assumption that $P \neq NP$, MIN (Π_1, Π_2)-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$ unless both Π_1 and Π_2 are classes of edgeless graphs.*

Note that if both Π_1 and Π_2 are classes of edgeless graphs, MIN (Π_1, Π_2)-PARTITION is solvable in polynomial time, because the problem is equivalent to 2-COLORING.

We can construct an approximation-preserving reduction from MIN (Π_1, Π_2)-PARTITION to MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION for any fixed $c \geq 3$, and obtain the following corollary.

Corollary 1. *Let $c \geq 3$ be a fixed constant, and let $\Pi_1, \Pi_2, \ldots, \Pi_c$ be any fixed nontrivial additive hereditary graph properties. Let G be a promised graph of n vertices, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Under the assumption that $P \neq NP$, MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$.*

We also study MIN (Π_1, Π_2)-PARTITION for planar bipartite graphs. Notice that any bipartite graph G has a (Π_1, Π_2)-coloring (i.e., $\mathsf{OPT}(G) \neq +\infty$) if both properties Π_1 and Π_2 are nontrivial, additive and hereditary.

Theorem 2. *Let Π_1 and Π_2 be any two fixed nontrivial additive hereditary graph properties, each of which contains a minimal forbidden induced subgraph that is planar and bipartite. Let G be a planar bipartite graph of n vertices which is promised, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Under the assumption that $P \neq NP$, MIN (Π_1, Π_2)-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$ unless both Π_1 and Π_2 are classes of edgeless graphs.*

In contrast to Theorem 1, Theorem 2 cannot be generalized for $c \geq 3$. In fact, it always holds that $\mathsf{OPT}(G) = 0$ for any $c \geq 3$ and any bipartite graph G if $\Pi_1, \Pi_2, \ldots, \Pi_c$ are nontrivial additive hereditary properties, because G has a $(\Pi_2, \Pi_3, \ldots, \Pi_c)$-coloring.

Theorem 2 immediately yields the following corollary.

Corollary 2. *Let Π_1 and Π_2 be any two classes of graphs listed below:*

- *edgeless graphs,*
- *cluster graphs (P_3-free graphs),*
- *cographs (P_4-free graphs),*
- *star graphs,*
- *path graphs,*
- *acyclic graphs,*
- *outerplanar graphs,*
- *series-parallel graphs,*
- *interval graphs,*
- *chordal graphs, or*
- *graphs of bounded maximum degree.*

Let G be a planar bipartite graph of n vertices which is promised, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Then, under the assumption that $P \neq NP$, MIN (Π_1, Π_2)-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$ unless both Π_1 and Π_2 are classes of edgeless graphs.

We prove Theorems 1 and 2 by giving a gap-producing reduction from POSITIVE 1-IN-3-SAT. In this paper, however, we omit the details due to the page limitation.

4. FPT Algorithm

In this section, we focus on the fixed-parameter tractability of PARAMETERIZED (Π_1, Π_2)-PARTITION when the graph property Π_2 is the class of acyclic graphs.

4.1 Hereditary Properties

We first consider the case where the graph property Π_1 is hereditary.

Theorem 3. *Let Π_1 be any hereditary graph property, and let Π_2 be the class of acyclic graphs. Given a graph G and a nonnegative integer k, suppose that one can decide in $t(k)$ time whether a subgraph H with at most k vertices of G satisfies Π_1. Then, PARAMETERIZED (Π_1, Π_2)-PARTITION for G can be solved in $2^{O(k^2)}(t(k) + n + m)$ time, where n and m are the numbers of vertices and edges in G, respectively.*

In this subsection, we also prove that the running time above can be improved for bounded degeneracy graphs. A graph G is *d-degenerate* if any subgraph of G has a vertex of degree at most d. It is known that many graph classes have bounded degeneracy: for example, planar graphs, graphs of bounded maximum degree, and bounded treewidth graphs.

Theorem 4. *Let Π_1 be any hereditary graph property, and let Π_2 be the class of acyclic graphs. Given a d-degenerate graph G and a nonnegative integer k, suppose that one can decide in $t(k)$ time whether a subgraph H with at most k vertices of G satisfies Π_1. Then, PARAMETERIZED (Π_1, Π_2)-PARTITION for G can be solved in $2^{O(h(k,d))}(t(k) + n + m)$ time, where $h(k, d) = \max\{d^3 + 3d^2 + 3d, (d + 1)\log k + \log(d + 1)\} \cdot k$, and n and m are the numbers of vertices and edges in G, respectively.*

For many natural properties, one can decide in $k^{O(1)}$ or $2^{O(k)}$ time whether a subgraph H with at most k vertices satisfies Π_1: for example, the classes of edgeless graphs, planar graphs, and proper c-colorable graphs for a fixed integer c. Thus, PARAMETERIZED (Π_1, Π_2)-PARTITION is solvable in $2^{O(k^2)}(n + m)$ time for general graphs and in $2^{O(k \log k)}(n + m)$ time for bounded degeneracy graphs, when Π_1 is such a natural hereditary property and Π_2 is the class of acyclic graphs.

To prove Theorems 3 and 4, we use the idea of a *compact representation* of minimal feedback vertex sets [4], [13]. Recall that a feedback vertex set S of a graph G is a vertex subset of G such that $G - S$ is acyclic. A compact representation for a set of minimal feedback vertex sets of a graph G is a set C of pairwise disjoint subsets of $V(G)$ such that choosing exactly one vertex from every set in C results in a minimal feedback vertex set of G. We say that a minimal feedback vertex set S of G *is contained in* a compact representation C if S can be obtained from C by this operation. A compact representation C is called a *k-compact representation* if the number of sets in C is at most k. We can efficiently enumerate k-compact representations of minimal feedback vertex sets in G, as follows:

Theorem 5 ([13]). *Given a graph G with m edges and an integer k, there exists an algorithm which enumerates k-compact representations of G in $O(23.1^k m)$ time such that any minimal feedback* *vertex set of size at most k is contained in some k-compact representation. Moreover, the number of k-compact representations output by the algorithm is at most $O(23.1^k)$.*

An instance (G, k) of PARAMETERIZED (Π_1, Π_2)-PARTITION is a yes-instance if and only if there is a (Π_1, Π_2)-coloring f of G such that $f^{-1}(1)$ forms a minimal feedback vertex set of size at most k of G, because Π_1 is hereditary. Therefore, PARAMETERIZED (Π_1, Π_2)-PARTITION can be rephrased as the problem of asking whether there exists a minimal feedback vertex set S of G such that $|S| \le k$ and $G[S]$ satisfies Π_1. A compact representation $\mathcal{C}$ is called *good* if $\mathcal{C}$ contains such a minimal feedback vertex set S. Given a graph and a k-compact representation $\mathcal{C}$, one can determine whether $\mathcal{C}$ is good or not, by the following lemma.

Lemma 1. *Let G be a graph with m edges. Given a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in G, assume that each set in $\mathcal{C}$ has at most α vertices. Then, one can determine whether $\mathcal{C}$ is good in $O(\alpha^k(t(k) + m))$ time under the assumption that one can decide in $t(k)$ time whether a subgraph H with at most k vertices of G satisfies Π_1.*

Proof. We enumerate all minimal feedback vertex sets of G contained in $\mathcal{C}$. Since $\mathcal{C}$ has at most k sets and each set in $\mathcal{C}$ has at most α vertices, $\mathcal{C}$ contains at most α^k minimal feedback vertex sets. For each minimal feedback vertex set S contained in $\mathcal{C}$, we construct $G[S]$ in $O(m)$ time and confirm that $G[S]$ satisfies Π_1 in $t(k)$ time. Therefore, we can determine whether $\mathcal{C}$ is good in $O(\alpha^k(t(k) + m))$ time. □

Therefore, our strategy is to enumerate k-compact representations of minimal feedback vertex sets in G by Theorem 5, and then check whether each enumerated k-compact representation $\mathcal{C}$ is good. Note that, however, the number α of vertices of each set in $\mathcal{C}$ is not always bounded by a function of k. Therefore, we kernelize each enumerated k-compact representation $\mathcal{C}$ to prove Theorems 3 and 4.

We now explain how to kernelize a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in G. A set in $\mathcal{C}$ is said to be *singleton* if the set consists of exactly one vertex, otherwise *multiple*. Then, the following proposition holds.

Proposition 1 ([4])**.** *Let C_1 and C_2 be any two distinct multiple sets in a compact representation $\mathcal{C}$ of minimal feedback vertex sets in a graph G. Then, any two vertices $v_1 \in C_1$ and $v_2 \in C_2$ are not adjacent in G.*

Let X be the set of the vertices of all singleton sets in $\mathcal{C}$. For a multiple set C in $\mathcal{C}$ and a subset $X' \subseteq X$, let $C_{X'}$ be the subset of C such that $N(u, X) = X'$ holds (on G) for every vertex u in $C_{X'}$. We iterate the following reduction rule for $\mathcal{C}$ until the rule is not applicable.

Reduction Rule. If there is a multiple set C in $\mathcal{C}$ such that $|C_{X'}| \ge 2$ for some $X' \subseteq X$, then choose an arbitrary vertex u from $C_{X'}$ and remove all vertices of $C_{X'} \setminus \{u\}$ from C.

Lemma 2. *Let $\mathcal{C}$ be a k-compact representation of minimal feedback vertex sets in a graph G. By applying Reduction Rule to $\mathcal{C}$, one can obtain a k-compact representation $\mathcal{C}^*$ of minimal feedback vertex sets in G such that*

(*a*) *each set in $\mathcal{C}^*$ has at most 2^k vertices of G; and*
(*b*) *$\mathcal{C}$ is good if and only if $\mathcal{C}^*$ is good.*

Proof. We first prove the claim (a). Suppose that $\mathcal{C}$ has a multiple set C with at least $2^k + 1$ vertices. Since $|X| \le k$, two vertices $u, u' \in C$ exist such that $N(u, X) = N(u', X)$ on G. Then, we apply Reduction Rule to C and obtain another k-compact representation. Thus, we can obtain a k-compact representation $\mathcal{C}^*$ such that each set in $\mathcal{C}^*$ has at most 2^k vertices by iterating Reduction Rule.

We next prove the claim (b). Let $\mathcal{C}'$ be a k-compact representation of G obtained by applying Reduction Rule to $\mathcal{C}$ once. It suffices to show that $\mathcal{C}$ is good if and only if $\mathcal{C}'$ is good. The if direction is straightforward, namely, if $\mathcal{C}'$ is good, then $\mathcal{C}$ is also good. We thus prove the only-if direction. Suppose that $\mathcal{C}$ is good, and let S be a minimal feedback vertex set of G such that S is contained in $\mathcal{C}$ and $G[S]$ satisfies Π_1. If $u \in S$, then $\mathcal{C}'$ also contains S and hence $\mathcal{C}'$ is good. Therefore, we suppose that $u \notin S$ and S has a vertex u' in $C_{X'} \setminus \{u\}$. Let $S' = (S \cup \{u\}) \setminus \{u'\}$. Then, S' is contained in $\mathcal{C}$, because u and u' are in the same set C in $\mathcal{C}$. Thus, S' is also contained in $\mathcal{C}'$. Moreover, from Proposition 1 and the assumption that $N(u, X) = N(u', X)$ holds, $G[S']$ is isomorphic to $G[S]$. Therefore, $G[S']$ satisfies Π_1, and hence $\mathcal{C}'$ is good. □

Proof of Theorem 3. Let (G, k) be an instance of PARAMETERIZED (Π_1, Π_2)-PARTITION, and let $n = |V(G)|$ and $m = |E(G)|$. Using Theorem 5, we first enumerate k-compact representations of all minimal feedback vertex sets in G in $O(23.1^k m)$ time. We then apply Reduction Rule to all enumerated k-compact representations. For each k-compact representation $\mathcal{C}$, by Lemma 2 we obtain a kernelized k-compact representation $\mathcal{C}^*$ such that each set in $\mathcal{C}^*$ has at most 2^k vertices of G; this can be done in $O(2^k kn + m)$ time. For each kernelized k-compact representation $\mathcal{C}^*$, by Lemma 1 we decide whether $\mathcal{C}^*$ is good in $O(2^{k^2} \cdot (t(k) + m))$ time. Theorem 5 says that there are at most $O(23.1^k)$ k-compact representations of G, and hence we produce kernelized k-compact representations in $O(23.1^k \cdot (2^k kn + m))$ time in total and determine whether there is a good k-compact representation of G in $O(23.1^k \cdot 2^{k^2} \cdot (t(k) + m))$ time in total. Therefore, the total running time of the algorithm is $2^{O(k^2)}(t(k) + n + m)$. This completes the proof of Theorem 3. □

We then prove Theorem 4. Suppose that a given graph G is d-degenerate for some integer $d \ge 1$. We apply the same algorithm (and hence the same Reduction Rule) to G. Using the fact that G is d-degenerate, we can estimate the size of each set in a kernelized compact representation more sharply, as follows.

Lemma 3. *Suppose that a graph G is d-degenerate for some integer $d \ge 1$. Let $\mathcal{C}$ be a k-compact representation of minimal feedback vertex sets in G. By applying Reduction Rule to $\mathcal{C}$, one can obtain a k-compact representation $\mathcal{C}^*$ of minimal feedback vertex sets in G such that*

(*a*) *each set in $\mathcal{C}^*$ has at most $2^{d^3+3d^2+3d}$ vertices of G if $k \le d^3 + 3d^2 + 3d$, otherwise it has less than $\sum_{i=0}^{d+1} \binom{k}{i}$ vertices of G; and* (*b*) *C is good if and only if C^* is good.*

Proof. We apply Reduction Rule to $\mathcal{C}$ until Reduction Rule is not applicable, and let $\mathcal{C}^*$ be the resulting compact representation. We here prove only the statement (a) because the statement (b) has been proved in the proof of Lemma 2.

We first consider the case that $k \le d^3 + 3d^2 + 3d$. In this case, for each set C in $\mathcal{C}^*$, it holds that $|C| \le 2^k \le 2^{d^3+3d^2+3d}$ by the same proof as that of Lemma 2.

Next, we consider the case that $k > d^3 + 3d^2 + 3d$. Assume for a contradiction that $\mathcal{C}^*$ has a multiple set C with at least $\sum_{i=0}^{d+1} \binom{k}{i}$ vertices. Let $w_1, w_2, \dots$ be the vertices of C in the non-increasing order of degree $d(w_i, X)$ on G. We pick the first $\sum_{i=0}^{d+1} \binom{k}{i}$ vertices on the order, and we denote by W a set of the vertices. Consider a bipartite graph $G' = (W \cup X, E)$, where $E = \{wx \in E(G) : w \in W \wedge x \in X\}$.

We calculate the value $|E| - d|W \cup X|$ to lead a contradiction. For every d-degenerate graph H, it holds that $|E(H)| \le d|V(H)|$. This can be shown inductively as follows. If $|V(H)| = 1$, it is trivial. If $|V(H)| > 1$, we pick a vertex v with at most degree d. Then, it holds that $|E(H)| \le |E(H - \{v\})| + d \le d|V(H - \{v\})| + d = d|V(H)|$. Therefore, since G' is a subgraph of G and hence G' is a d-degenerate graph, we have $|E| - d|W \cup X| \le 0$.

On the other hand, we also show that we have $|E| - d|W \cup X| > 0$. Obviously, we have $|W \cup X| \le \sum_{i=0}^{d+1} \binom{k}{i} + k$. Moreover, it holds that $|E| \ge \sum_{i=0}^{d+1} i \cdot \binom{k}{i}$, because there is at most $\binom{k}{i}$ vertices of degree i in W by Reduction Rule. Thus, we have

$$
\begin{aligned}
|E| - d|W \cup X| &\ge \sum_{i=0}^{d+1} i \cdot \binom{k}{i} - d\left(\sum_{i=0}^{d+1} \binom{k}{i} + k\right) \\
&= \sum_{i=0}^{d+1} (i - d) \cdot \binom{k}{i} - dk \\
&= \sum_{i=0}^{d} (i - d) \cdot \binom{k}{i} + \binom{k}{d+1} - dk \\
&\ge -\sum_{i=0}^{d} d \cdot \binom{k}{d} + \binom{k}{d+1} - dk \\
&= -d(d+1) \cdot \binom{k}{d} + \frac{k-d}{d+1}\binom{k}{d} - dk \\
&= \left(-d(d+1) + \frac{k-d}{d+1}\right)\binom{k}{d} - dk.
\end{aligned}
$$

From the assumption that $k > d^3 + 3d^2 + 3d$, we have

$$|E| - d|W \cup X| > d\binom{k}{d} - dk \ge d\binom{k}{1} - dk \ge 0.$$

This completes the proof of Lemma 3. □

Proof of Theorem 4. By Lemma 3, we can obtain a k-compact representation $\mathcal{C}^*$ of a d-degenerate graph G such that each set in $\mathcal{C}^*$ has at most $\max\{2^{d^3+3d^2+3d}, 2^{(d+1)\log k + \log(d+1)}\}$ vertices in $O(2^k kn + m)$ time from a given k-compact representation of G. Combined with Lemma 1, we decide whether a k-compact representation of G is good in $O(2^{h(k,d)} \cdot (t(k) + m))$ time, where $h(k,d) = \max\{d^3+3d^2+3d, (d+1)\log k + \log(d+1)\} \cdot k$. Theorem 5 says that there are at most $O(23.1^k)$ k-compact representations of G, and hence we produce kernelized k-compact representations in $O(23.1^k \cdot (2^k kn + m))$ time in total and determine whether there is a good k-compact representation of G in $O(23.1^k \cdot 2^{h(k,d)} \cdot (t(k) + m))$ time in total. Therefore, the total running time of the algorithm is $2^{O(h(k,d))}(t(k) + n + m)$. This completes the proof of Theorem 4. □

4.2 Graph Properties with Bounded Maximum Degree

The parameterized variant of Independent Feedback Vertex Set is equivalent to Parameterized (Π_1, Π_2)-Partition when Π_1 is the class of edgeless graphs and Π_2 is the class of acyclic graphs. Since the class of edgeless graphs is the class of graphs with maximum degree zero, it is natural to consider the case where Π_1 is the class of graphs with bounded maximum degree. In this subsection, we give the following theorem for such a case.

Theorem 6. *Let Π_1 be the class of graphs with maximum degree Δ for a fixed integer Δ, and let Π_2 be the class of acyclic graphs. Given a graph G with n vertices and m edges,* Parameterized (Π_1, Π_2)-Partition *can be solved in $O(23.1^k m) + 2^{O(\Delta k \log k)}(n + m)$ time.*

Our algorithm for Theorem 6 takes a similar strategy as in Section 4.1, but employs the following modified reduction rule to kernelize a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in a graph G. We iterate each reduction rule for $\mathcal{C}$ until the rule is not applicable. Recall that X denotes the set of the vertices of all singleton sets in $\mathcal{C}$.

Modified Reduction Rule.

Rule A: if there is a multiple set C in $\mathcal{C}$ containing a vertex u such that $|N(u, X)| \ge \Delta + 1$, then remove u from C; and

Rule B: if there is a multiple set C in $\mathcal{C}$ such that $|C_{X'}| \ge 2$ for some $X' \subseteq X$, then choose an arbitrary vertex u from $C_{X'}$ and remove all vertices of $C_{X'} \setminus \{u\}$ from C.

The correctness of Rule A is clear because $G[X \cup \{u\}]$ does not satisfy Π_1, and the correctness of Rule B has been proved in the proof of Lemma 2.

Proof of Theorem 6. We first enumerate k-compact representations of all minimal feedback vertex sets in G in $O(23.1^k m)$ time by Theorem 5. For each k-compact representation $\mathcal{C}$, we obtain a kernelized k-compact representation $\mathcal{C}^*$ such that each set in $\mathcal{C}^*$ has at most Δk^{Δ} vertices of G because there are at most $\binom{k}{i}$ vertices of degree i, where $0 \le i \le \Delta$, in C after applying Rule A and Rule B. This can be done in $O(2^k kn + m)$ time in total. Since any graph H with at most k vertices can be checked in $O(\Delta k)$ time whether H satisfies Π_1, for each kernelized k-compact representation $\mathcal{C}^*$, we decide whether $\mathcal{C}^*$ is good in $O((\Delta k^{\Delta})^k \cdot (\Delta k + m))$ time by Lemma 1. Therefore, the total running time of the algorithm is $O(23.1^k m) + 2^{O(\Delta k \log k)}(n + m)$ time in total. This completes the proof of Theorem 6. □

Although one can obtain the faster FPT algorithm from Theorem 6 when Δ is a constant, its running time does not achieve a single exponential even if $\Delta = 1$. For this reason, we give a single exponential FPT algorithm when $\Delta = 1$.

Theorem 7. *Let Π_1 be a class of graphs with maximum degree one and let Π_2 be a class of acyclic graphs. Then,* Parameterized (Π_1, Π_2)-Partition *can be solved in $O(23.1^k(k^{2.5} + n + m))$ time.*

Given an instance (G, k) of Parameterized (Π_1, Π_2)-Partition, our algorithm first enumerates all k-compact representations of G. For each k-compact representation C, we apply the following four reduction rules from top to bottom.

Reduction Rule 1. If $G[X]$ does not satisfy Π_1, then determine that a k-compact representation is not good.

Reduction Rule 2. If there is a vertex u of a multiple set C in a k-compact representation such that $N(u, X) = \emptyset$, then remove all vertices of $C \setminus \{u\}$ from C. This reduction rule is iterated until it is not applicable.

Reduction Rule 3. If there is a vertex u of a multiple set C in a k-compact representation such that $G[X \cup \{u\}]$ does not satisfies Π_1, then remove u from C. This reduction rule is iterated until it is not applicable.

Reduction Rule 4. Construct a bipartite graph $B = (W \cup X, E)$ from a k-compact representation C, where each vertex $w \in W$ corresponds to a multiple set $C_w \in C$. A vertex $x \in X$ and a vertex $w \in W$ are joined by an edge if and only if a multiple set C_w corresponding w has a vertex u such that $N(u, X) = \{x\}$ on G. Then, compute a maximum matching M of B. If $|M| = |W|$, determine that C is good, otherwise C is not good.

The correctness of Reduction Rules 1 and 3 are straightforward. We show that the correctness of Reduction Rules 2 and 4.

Lemma 4. *Reduction Rule 2 is correct.*

Proof. Let C' be a k-compact representation of G obtained by applying Reduction Rule 2 to C once. It suffices to show that C is good if and only if C' is good. The if direction is straightforward, namely, if C' is good, then C is also good. We thus prove the only-if direction. Suppose that C is good, and let S be a minimal feedback vertex set of G such that S is contained in C and $G[S]$ satisfies Π_1. If $u \in S$, then C' also contains S and hence C' is good. We suppose that $u \notin S$ and S has a vertex $u' \in C \setminus \{u\}$. Let $S' = (S \cup \{u\}) \setminus \{u'\}$. Then, S' is contained in C, because u and u' are in the same set C in C. Thus, S' is also contained in C'. Moreover, from Proposition 1 and the assumption that $N(u, X) = \emptyset$ holds, u is an isolated vertex on $G[S']$. Therefore, since $G[S]$ satisfies Π_1, $G[S']$ satisfies Π_1 and hence C' is good. □

Lemma 5. *In Reduction Rule 4, there is a matching M of a bipartite graph $B = (W \cup X, E)$ with size exactly $|W|$ if and only if C is good.*

Proof. We first show the if direction. Suppose that S be a minimal feedback vertex set of G such that S is contained in C and $G[S]$ satisfies Π_1. Let $S' = S \setminus X$. From Reduction Rule 2 and the assumption that $G[S]$ satisfies Π_1, each vertex in S' has degree exactly one on $G[S]$. Moreover, edges incident to a vertex in S' do not share an endpoint; otherwise, $G[S]$ has a vertex with degree at least two. Therefore, we can construct a matching M' of G with size exactly $|S'|$ such that every vertex in S' is an endpoint of some edge in M'. Since all vertices in S' are contained in distinct multiple sets of C and each multiple set corresponds to a vertex of W in a bipartite graph B, we can also obtain a matching M of B with size exactly $|W|$ from M'.

We next show the only-if direction. For each edge $wx \in M$ such that $w \in W$ and $x \in X$, there is a vertex u in a multiple set C_w corresponding w such that $N(u, X) = \{x\}$ on G from the definition. Let S' be a set of such a vertex u for each edge in M, and let $S = S' \cup X$. Since each vertex in S is chosen from each set in C, the vertex set S forms a minimal feedback vertex set of G. To show that C is good, we prove that $G[S]$ satisfies Π_1. Each vertex u in S' has degree exactly one on $G[S]$ from the fact that $N(u, X) = \{x\}$ and Proposition 1. Any vertex in X that is not an endpoint of an edge in M has degree at most one on $G[S]$ because Reduction Rule 1 is not applicable to C. Moreover, any vertex $x \in X$ that is an endpoint of some edge in M has degree exactly one on $G[S]$ as follows. x has degree at most one on $G[X]$ from Reduction Rule 1. If x has degree zero on $G[X]$, since there exists exactly one edge in M that has x as an endpoint, x has degree exactly one on $G[S]$. If x has degree one on $G[X]$, there exists no vertex in W that is adjacent to x from Reduction Rule 3 and hence x has degree exactly one on $G[S]$. As a conclusion, every vertex in $G[S]$ has degree at most one, that is, $G[S]$ satisfies Π_1. This completes the proof of Lemma 5. □

Finally, we estimate the running time of our algorithm. All k-compact representations of G are enumerated in $O(23.1^k m)$ time by Theorem 5. For each k-compact representation, Reduction Rules 1-3 can be executed in $O(n + m)$ time. In Reduction Rule 4, a bipartite graph B is constructed in $O(m)$ time. Since B has at most k vertices and at most k^2 edges, a maximal matching of B can be computed in $O(k^{2.5})$ by Hopcroft-Karp algorithm [5]. Theorem 5 says that there are at most $O(23.1^k)$ k-compact representations of G, and hence the total running time is $O(23.1^k m + 23.1^k(k^{2.5} + n + m)) = O(23.1^k(k^{2.5} + n + m))$. This completes the proof of Theorem 7. □

Acknowledgments

Yuma Tamura is partially supported by JSPS KAKENHI Grant Number JP20J11259, Japan. Takehiro Ito is partially supported by JSPS KAKENHI Grant Numbers JP18H04091, JP19K11814 and JP20H05793, Japan. Xiao Zhou is partially supported by JSPS KAKENHI Grant Number JP19K11813, Japan.

References

[1] Alekseev, V. E., Farrugia, A. and Lozin, V. V.: New Results on Generalized Graph Coloring, *Discrete Mathematics and Theoretical Computer Science*, Vol. 6, No. 2, pp. 215–222 (online), available from ⟨http://dmtcs.episciences.org/311⟩ (2004).

[2] Bafna, V., Berman, P. and Fujito, T.: A 2-Approximation Algorithm for the Undirected Feedback Vertex Set Problem, *SIAM Journal on Discrete Mathematics*, Vol. 12, No. 3, pp. 289–297 (online), DOI: 10.1137/S0895480196305124 (1999).

[3] Farrugia, A.: Vertex-Partitioning into Fixed Additive Induced-

Hereditary Properties Is NP-hard, *Electronic Journal of Combinatorics*, Vol. 11, p. R46 (online), DOI: https://doi.org/10.37236/1799 (2004).

[4] Guo, J., Gramm, J., Hüffner, F., Niedermeier, R. and Wernicke, S.: Compression-based fixed-parameter algorithms for feedback vertex set and edge bipartization, *Journal of Computer and System Sciences*, Vol. 72, No. 8, pp. 1386–1396 (online), DOI: https://doi.org/10.1016/j.jcss.2006.02.001 (2006).

[5] Hopcroft, J. and Karp, R.: An $n^{5/2}$ Algorithm for Maximum Matchings in Bipartite Graphs, *SIAM Journal on Computing*, Vol. 2, No. 4, pp. 225–231 (online), DOI: 10.1137/0202019 (1973).

[6] Iwata, Y. and Kobayashi, Y.: Improved Analysis of Highest-Degree Branching for Feedback Vertex Set, *14th International Symposium on Parameterized and Exact Computation, IPEC 2019, September 11-13, 2019, Munich, Germany* (Jansen, B. M. P. and Telle, J. A., eds.), LIPIcs, Vol. 148, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, pp. 22:1–22:11 (online), DOI: 10.4230/LIPIcs.IPEC.2019.22 (2019).

[7] Kanj, I., Komusiewicz, C., Sorge, M. and van Leeuwen, E. J.: Parameterized algorithms for recognizing monopolar and 2-subcolorable graphs, *Journal of Computer and System Sciences*, Vol. 92, pp. 22–47 (online), DOI: https://doi.org/10.1016/j.jcss.2017.08.002 (2018).

[8] Li, S. and Pilipczuk, M.: An Improved FPT Algorithm for Independent Feedback Vertex Set, *Graph-Theoretic Concepts in Computer Science* (*WG 2018*), pp. 344–355 (online), DOI: 10.1007/978-3-030-00256-5_28 (2018).

[9] Lokshtanov, D.: Wheel-Free Deletion Is W[2]-Hard, *Parameterized and Exact Computation, Third International Workshop* (*IWPEC 2008*), pp. 141–147 (online), DOI: 10.1007/978-3-540-79723-4_14 (2018).

[10] Luo, J., Molter, H. and Suchý, O.: A Parameterized Complexity View on Collapsing *k*-Cores, *13th International Symposium on Parameterized and Exact Computation (IPEC 2018)* (Paul, C. and Pilipczuk, M., eds.), Leibniz International Proceedings in Informatics (LIPIcs), Vol. 115, Dagstuhl, Germany, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, pp. 7:1–7:14 (online), DOI: 10.4230/LIPIcs.IPEC.2018.7 (2019).

[11] Marx, D., O'sullivan, B. and Razgon, I.: Finding Small Separators in Linear Time via Treewidth Reduction, *ACM Transactions on Algorithms*, Vol. 9, No. 4 (online), DOI: 10.1145/2500119 (2013).

[12] Misra, N., Philip, G., Raman, V. and Saurabh, S.: On Parameterized Independent Feedback Vertex Set, *Theoretical Computer Science*, Vol. 461, pp. 65–75 (online), DOI: https://doi.org/10.1016/j.tcs.2012.02.012 (2012).

[13] Misra, N., Philip, G., Raman, V., Saurabh, S. and Sikdar, S.: FPT algorithms for Connected Feedback Vertex Set, *Journal of Combinatorial Optimization*, Vol. 24, No. 2, pp. 131–146 (online), DOI: 10.1007/s10878-011-9394-2 (2012).

[14] Tamura, Y., Ito, T. and Zhou, X.: Approximability of the Independent Feedback Vertex Set Problem for Bipartite Graphs, *WALCOM: Algorithms and Computation - 14th International Conference* (*WALCOM 2020*), pp. 286–295 (online), DOI: 10.1007/978-3-030-39881-1_24 (2020).

