

Technical Note

Dosun-Fuwari is NP-complete

CHUZO IWAMOTO^{1,a)} TATSUAKI IBUSUKI^{2,b)}

Received: October 20, 2017, Accepted: December 8, 2017

Abstract: Dosun-Fuwari is one of Nikoli's pencil puzzles, which is played on a rectangular grid of cells. Some of the cells are colored black, and the remaining cells are divided into rooms. The purpose of the puzzle is to place balloons and iron balls according to the following rules: Place one balloon and one iron ball in each room. Balloons (resp. iron balls) are light and float (heavy and sink), so they must be placed in the top (bottom) row of the grid of cells, or in a cell right under (over) a black cell or right under other balloons (over other iron balls). It is shown that deciding whether a Dosun-Fuwari puzzle has a solution is NP-complete.

Keywords: Dosun-Fuwari, pencil puzzle, NP-complete

1. Introduction

The Dosun-Fuwari puzzle is played on a rectangular grid of cells (see Fig. 1 (a)). Initially, some of the cells are colored black, and the remaining cells are divided into rooms surrounded by bold lines. The purpose of the puzzle is to place *balloons* (white circles) and *iron balls* (black circles) (see Fig. 1 (f)) according to the following rules [13]: (1) Place one balloon and one iron ball in each of the rooms. (2) Balloons are light and float, so they must be placed in the top row of the grid of cells, or in a cell right under a black cell or right under other balloons. (3) Iron balls are heavy and sink, so they must be placed in the bottom row of the grid of cells, or in a cell right over a black cell or right over other iron balls.

Figure 1 (a) is the initial configuration of a Dosun-Fuwari puzzle. In the figure, 6×6 cells are divided into nine rooms and five black cells. From Fig. 1 (b)–(e), the reader can understand basic techniques for finding a solution.

In Fig. 1 (b), consider the 3×1 room in the third column. Iron balls are heavy and sink, and balloons are light and float; therefore, balloon 1 and iron ball 2 are placed in the top and bottom of the room, respectively. Then, a balloon must be placed in cell 3 (which is right over balloon 1) so that balloon 1 does not float (see balloon 3 in Fig. 1 (c)). Similarly, two iron balls must be placed in cells 4 and 5 of Fig. 1 (b) so that iron ball 2 does not sink (see iron balls 4 and 5 in Fig. 1 (c)). In Fig. 1 (c), two iron balls and a balloon are placed in the cells 6, 7 and 8, respectively.

In Fig. 1 (d), there is a 2×1 yellow area in the fifth column of the grid (see cells 8 and 3); therefore, a balloon cannot be placed in cell 1, 2, or 3. Hence, two balloons are placed in cells 4 and 5. Since there is no iron ball in cell 1, an iron ball cannot be placed in cell 3. Thus, two iron balls are placed in cells 6 and 7, and balloons are placed in cells 8 and 9.

In Fig. 1 (e), an iron ball can be placed in one of the cells 1 and 2; in the figure, it is placed in cell 1. Finally, balloons and an iron ball are placed in cell 3, 4 and 5, respectively. Figure 1 (f) is a solution.

In this paper, we study the computational complexity of the decision version of the Dosun-Fuwari puzzle. The instance of the *Dosun-Fuwari puzzle problem* is defined as a rectangular grid of cells, which are partitioned into rooms and black cells. The problem is to decide whether there is a solution to the instance. It is shown that the Dosun-Fuwari puzzle problem is NP-complete. It is clear that the problem belongs to NP, since the game ends when every room contains one balloon and one iron ball.

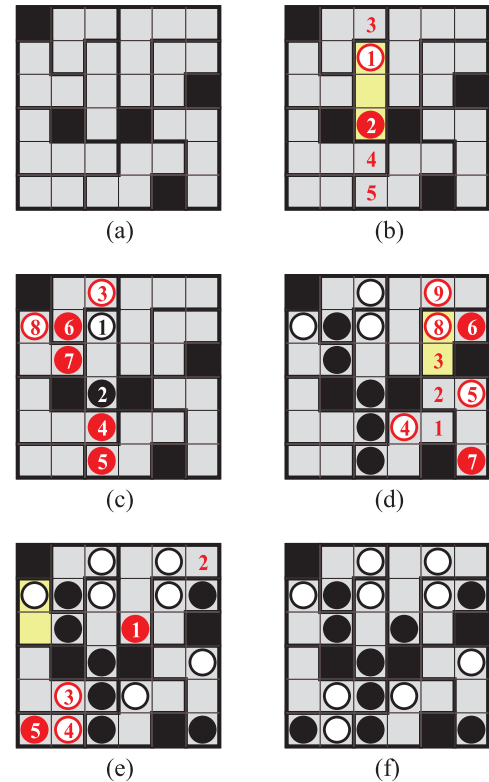


Fig. 1 (a) Initial configuration of a Dosun-Fuwari puzzle. (b)–(f) are the progress from the initial configuration to a solution.

¹ Graduate School of Engineering, Hiroshima University, Higashi-hiroshima, Hiroshima 739-8521, Japan

² School of Integrated Arts and Sciences, Hiroshima University, Higashi-hiroshima, Hiroshima 739-8521, Japan

^{a)} chuzo@hiroshima-u.ac.jp

^{b)} b154906@hiroshima-u.ac.jp

There has been a huge amount of literature on the computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [6]. After the publication of this book, the following Nikoli's pencil puzzles were shown to be NP-complete: Fillmat [11], Hashiwokakero [2], Kurodoko [9], Numberlink [1], Pipe Link [12], Shakashaka [4], Shikaku and Ripple Effect [10], Yajilin and Country Road [7], and Yosenabe [8].

2. NP-completeness of Dosun-Fuwari

We present a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a Dosun-Fuwari puzzle such that C is satisfiable if and only if the puzzle has a solution.

2.1 PLANAR 3SAT Problem

The definition of PLANAR 3SAT is mostly from Ref. [5]. Let $U = \{x_1, x_2, \dots, x_n\}$ be a set of Boolean variables. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U , then x and $\bar{x}$ are *literals* over U . The value of $\bar{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U , such as $\{\bar{x}_1, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \dots, c_m\}$ of clauses over U such that (i) $|c_j| \leq 3$ for each $c_j \in C$ and (ii) the bipartite graph $G = (V, E)$, where $V = U \cup C$ and E contains exactly those pairs $\{x, c\}$ such that either literal x or $\bar{x}$ belongs to the clause c , is planar.

The PLANAR 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C . This problem is known to be NP-complete. For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\bar{x}_1, \bar{x}_2, \bar{x}_4\}$, $c_3 = \{\bar{x}_1, \bar{x}_3, x_4\}$, $c_4 = \{\bar{x}_2, \bar{x}_3, \bar{x}_4\}$ provide an instance of PLANAR 3SAT. For this instance, the answer is “yes,” since there is a truth assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfying all clauses. It is known that PLANAR 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [3].

2.2 Transformation from an Instance of PLANAR 3SAT to a Dosun-Fuwari Puzzle

Each variable $x_i \in \{x_1, x_2, \dots, x_n\}$ is transformed into the variable gadget as illustrated in Fig. 2, which consists of two yellow

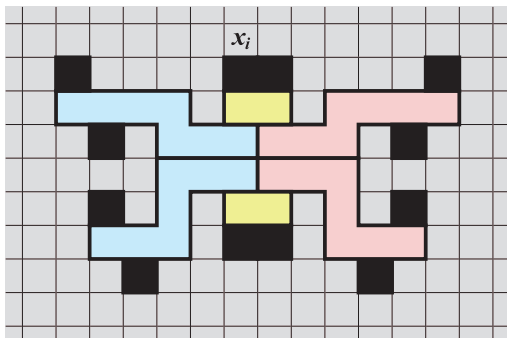
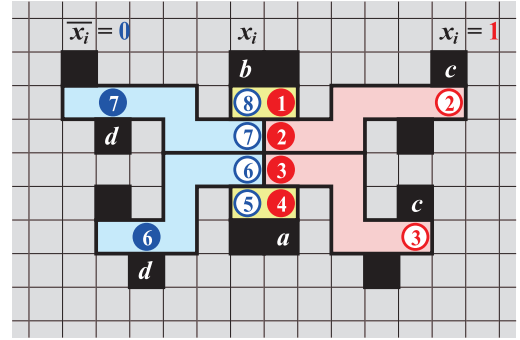


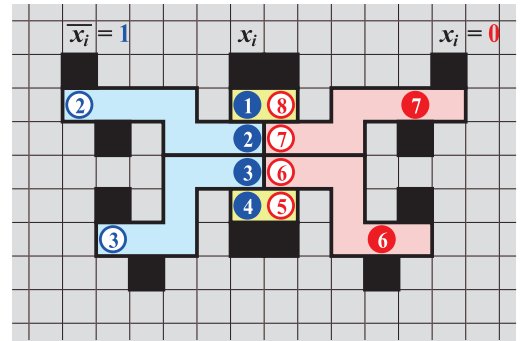
Fig. 2 Variable gadget transformed from x_i .

rooms, two blue rooms, two red rooms, and 12 black cells.

There are two possible solutions to this gadget. Suppose that the right cell of the upper yellow room has iron ball 1 (see Fig. 3(a)). Since iron ball 1 is heavy, three iron balls must be placed in cells 2, 3, and 4 (which are over black cell a) so that iron ball 1 does not sink. Then, the left cell of the lower yellow room must have a balloon (see balloon 5). Since balloon 5 is light, three balloons must be placed in cells 6, 7, and 8 (which are under black cell b) so that balloon 5 does not float. Finally, two

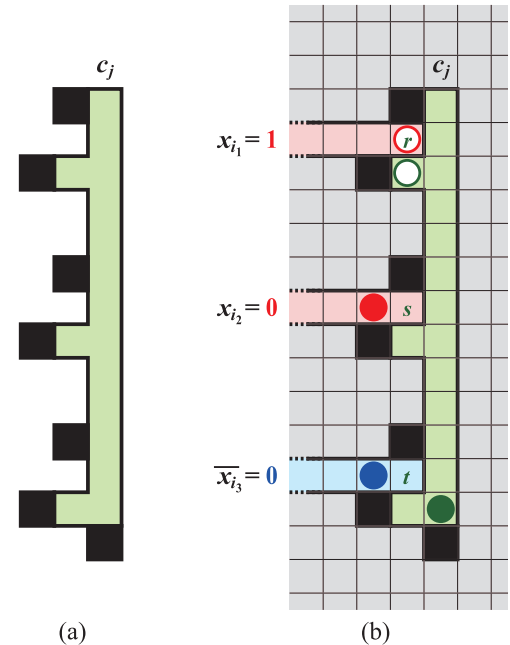


(a)



(b)

Fig. 3 (a) A solution when $x_i = 1$. (b) A solution when $\bar{x}_i = 1$.



(a)

(b)

Fig. 4 (a) Clause gadget transformed from c_j . (b) Suppose $c_j = \{x_{i_1}, x_{i_2}, \bar{x}_{i_3}\}$. If at least one of variables $x_{i_1}, x_{i_2}, \bar{x}_{i_3}$ is 1, then there is a solution to the green room.

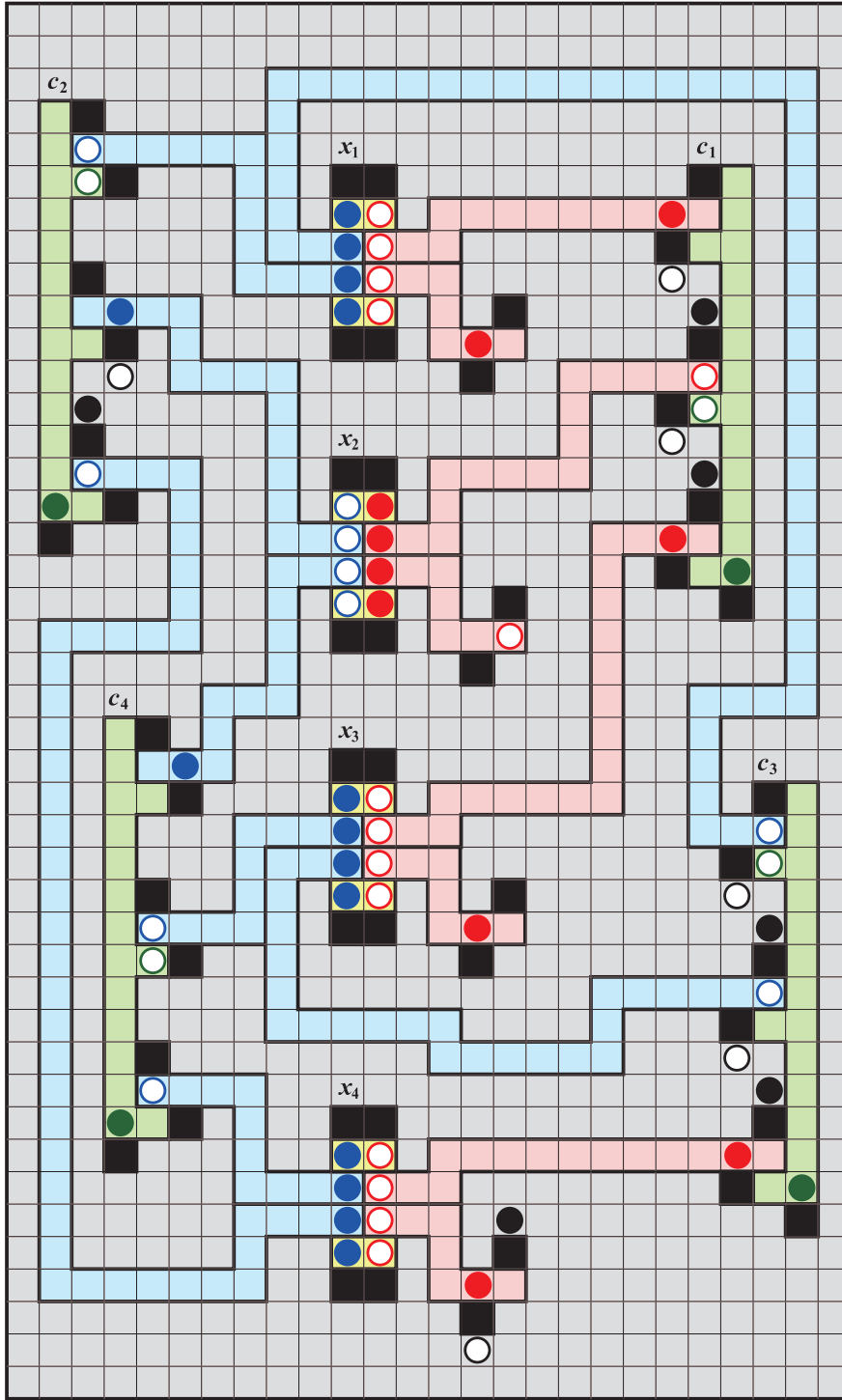


Fig. 5 A Dosun-Fuwari puzzle transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\bar{x}_1, \bar{x}_2, \bar{x}_4\}$, $c_3 = \{\bar{x}_1, \bar{x}_3, x_4\}$, $c_4 = \{\bar{x}_2, \bar{x}_3, \bar{x}_4\}$. From the solution of the puzzle, one can see that the assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfies all clauses of C .

balloons 2 and 3 are placed in the cells under black cells c , and two iron balls 6 and 7 in the cells over black cells d . This position of iron balls and balloons corresponds to $x_i = 1$.

On the other hand, suppose the left cell of the upper yellow room has iron ball 1 (see Fig. 3 (b)). By an observation similar to the previous paragraph, six iron balls and six balloons are placed in the cells indicated in Fig. 3 (b). This position of iron balls and balloons corresponds to $\bar{x}_i = 1$.

Clause $c_j \in \{c_1, c_2, \dots, c_m\}$ is transformed into the clause

gadget as illustrated in **Fig. 4(a)**, which is a green room with black cells. This gadget is connected to three variable gadget if c_j consists of three literals (see Fig. 4(b) and **5**). Suppose that $c_j = \{x_{i_1}, x_{i_2}, \bar{x}_{i_3}\}$ (see red and blue cells of Fig. 4(b)). In Fig. 4(b), at least one of the three cells r , s , and t has a balloon if and only if a balloon can be placed in the green room. Hence, one can see that at least one of c_j 's literals is 1 if and only if there is a solution to the green room.

Figure 5 is a Dosun-Fuwari puzzle transformed from $C =$

$\{c_1, c_2, c_3, c_4\}$ and $U = \{x_1, x_2, x_3, x_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. In this figure, gray cells are partitioned into six gray rooms. In each gray room, there is a balloon right under a black cell, and there is an iron ball right over a black cell. It should be noted that any balloon (resp. iron ball) placed in gray rooms cannot be used for placing balloons (resp. iron balls) in red, blue, or green rooms, since every balloon (resp. iron ball) placed in gray rooms is always right over (resp. right under) a gray empty cell.

From this construction, the instance C of PLANAR 3SAT is satisfiable if and only if the corresponding Dosun-Fuwari puzzle has a solution. The solution given in **Fig. 5** indicates that the assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfies all clauses of C .



Tatsuaki Ibusuki is an undergraduate student at the School of Integrated Arts and Sciences, Hiroshima University.

References

- [1] Adcock, A.B., Demaine, E.D., Demaine, M.L., O'Brien, M.P., Reidl, F., Villaamil, F.S. and Sullivan, B.D.: Zig-zag Numberlink is NP-complete, *J. Inf. Process.*, Vol.23, No.3, pp.239–245 (2015).
- [2] Andersson, D.: Hashiwokakero is NP-complete, *Inf. Process. Lett.*, Vol.109, pp.1145–1146 (2009).
- [3] Cerioli, M.R., Faria, L., Ferreira, T.O., Martinhon, C.A.J., Protti, F. and Reed, B.: Partition into Cliques for Cubic Graphs: Planar Case, Complexity and Approximation, *Discrete Appl. Math.*, Vol.156, No.12, pp.2270–2278 (2008).
- [4] Demaine, E.D., Okamoto, Y., Uehara, R. and Uno, Y.: Computational Complexity and an Integer Programming Model of Shakashaka, *Proc. 25th Canadian Conference on Computational Geometry* (2013) (online), available from <http://cccg.ca/> (accessed 2017-10-20).
- [5] Garey, M.R. and Johnson, D.S.: *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman, New York, NY, USA (1979).
- [6] Hearn, R.A. and Demaine, E.D.: *Games, Puzzles, and Computation*, A K Peters Ltd., MA, USA (2009).
- [7] Ishibashi, A., Sato, Y. and Iwata, S.: NP-completeness of Two Pencil Puzzles: Yajilin and Country Road, *Utilitas Mathematica*, Vol.88, pp.237–246 (2012).
- [8] Iwamoto, C.: Yosenabe is NP-complete, *J. Inf. Process.*, Vol.22, No.1, pp.40–43 (2014).
- [9] Kölker, J.: Kurodoko is NP-complete, *J. Inf. Process.*, Vol.20, No.3, pp.694–706 (2012).
- [10] Takenaga, Y., Aoyagi, S., Iwata, S. and Kasai, T.: Shikaku and Ripple Effect are NP-complete, *Congressus Numerantium*, Vol.216, pp.119–127 (2013).
- [11] Uejima, A. and Suzuki, H.: Fillmat is NP-complete and ASP-complete, *J. Inf. Process.*, Vol.23, No.3, pp.310–316 (2015).
- [12] Uejima, A., Suzuki, H. and Okada, A.: The complexity of generalized pipe link puzzles, *J. Inf. Process.*, Vol.25, pp.724–729 (2017).
- [13] WEB Nikoli, Dosun-Fuwari (online), available from <http://nikoli.co.jp/en/puzzles/dosunfuwari.html> (accessed 2017-10-20).



Chuzo Iwamoto received his B.Eng. degree from Yamaguchi University in 1990, and M.Eng. and Dr.Eng. degrees from Kyushu University in 1992 and 1995, respectively. From 1995 to 1997, he was a lecturer at Kyushu Institute of Design. In 1997, he joined Hiroshima University as an associate professor, and he is currently

a professor of the Graduate School of Engineering, Hiroshima University.