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Balanced (C_5, C_6)-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_5, C_6 be *the* 5-*cycle* and *the* 6-*cycle*, respectively. *The* (C_5, C_6)-$2t$-*foil* is a graph of t edge-disjoint C_5's and t edge-disjoint C_6's with a common vertex and the common vertex is called *the center of the* (C_5, C_6)-$2t$-*foil*.
When K_n is decomposed into edge-disjoint sum of (C_5, C_6)-$2t$-foils, we say that K_n *has a* (C_5, C_6)-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_5, C_6)-$2t$-foils, we say that K_n *has a balanced* (C_5, C_6)-$2t$-*foil decomposition* and this number is called *the replication number*.

2. Balanced (C_5, C_6)-$2t$-foil decomposition of K_n

Notation. We denote a (C_5, C_6)-$2t$-foil passing through
$1-2-3-4-5-1-6-7-8-9-10-1$,
$1-11-12-13-14-1-15-16-17-18-19-1$,
$1-20-21-22-23-1-24-25-26-27-28-1$,
...,
$1-(9t-7)-(9t-6)-(9t-4)-(9t-4)-1-(9t-3)-(9t-2)-(9t-1)-9t-(9t+1)-1$
by
$\{(1,2,3,4,5),(1,6,7,8,9,10)\}$
$\cup\ \{(1,11,12,13,14),(1,15,16,17,18,19)\}$
$\cup\ \{(1,20,21,22,23),(1,24,25,26,27,28)\}$
...
$\cup\ \{(1,9t-7,9t-6,9t-5,9t-4),(1,9t-3,9t-2,9t-1,9t,9t+1)\}$.

Theorem. K_n has a balanced (C_5, C_6)-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{22t}$.

Proof. **(Necessity)** Suppose that K_n has a balanced (C_5, C_6)-$2t$-foil decomposition. Let b be the number of (C_5, C_6)-$2t$-foils and r be the replication number. Then $b = n(n-1)/22t$ and $r = (9t+1)(n-1)/22t$. Among r (C_5, C_6)-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_5, C_6)-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/22t$ and $r_2 = 9(n-1)/22$. Therefore, $n \equiv 1 \pmod{22t}$ is necessary.

(Sufficiency) Put $n = 22st+1$, $T = st$.
Case 1. T = 1 and n = 23. (Example 1.)
$B_i = \{(i, i+5, i+8, i+19, i+9), (i, i+1, i+3, i+7, i+14, i+6)\}$ $(i = 1, 2, ..., 23)$.
Then they comprise a balanced (C_5, C_6)-2-foil decomposition of K_{23}.
Case 2. T $\geq$ 2 and n = 22T + 1.
$B_i = \{(i, i+4T+1, i+6T+2, i+16T+3, i+8T+1), (i, i+1, i+T+2, i+5T+2, i+10T+4, i+5T+1)\}$
$\cup\ \{(i, i+4T+2, i+6T+4, i+16T+7, i+8T+3), (i, i+2, i+T+4, i+5T+3, i+11T+7, i+6T+3)\}$
$\cup\ \{(i, i+4T+3, i+6T+6, i+16T+11, i+8T+5), (i, i+3, i+T+6, i+5T+4, i+11T+10, i+6T+5)\}$
...
$\cup\ \{(i, i+5T-2, i+8T-4, i+20T-9, i+10T-5), (i, i+T-2, i+3T-4, i+6T-1, i+14T-5, i+8T-5)\}$
$\cup\ \{(i, i+5T-1, i+8T-2, i+20T-5, i+10T-3), (i, i+T-1, i+3T-2, i+6T, i+14T-2, i+8T-3)\}$
$\cup\ \{(i, i+5T, i+8T, i+20T-1, i+10T-1), (i, i+T, i+3T, i+6T+1, i+14T+1, i+8T-1)\}$
$(i = 1, 2, ..., n)$.
Decompose each (C_5, C_6)-$2T$-foil into s (C_5, C_6)-$2t$-foils. Then they comprise a balanced (C_5, C_6)-$2t$-foil decomposition of K_n. ■

Example 2. A balanced $(\mathbf{C_5}, \mathbf{C_6})$-4-foil decomposition of $\mathbf{K_{45}}$.
$B_i = \{(i, i+9, i+14, i+35, i+17), (i, i+1, i+4, i+12, i+24, i+11)\}$
$\cup \{(i, i+10, i+16, i+39, i+19), (i, i+2, i+6, i+13, i+29, i+15)\}$
$(i = 1, 2, ..., 45)$.

Example 3. A balanced $(\mathbf{C_5}, \mathbf{C_6})$-6-foil decomposition of $\mathbf{K_{67}}$.
$B_i = \{(i, i+13, i+20, i+51, i+25), (i, i+1, i+5, i+17, i+34, i+16)\}$
$\cup \{(i, i+14, i+22, i+55, i+27), (i, i+2, i+7, i+18, i+40, i+21)\}$
$\cup \{(i, i+15, i+24, i+59, i+29), (i, i+3, i+9, i+19, i+43, i+23)\}$
$(i = 1, 2, ..., 67)$.

Example 4. A balanced $(\mathbf{C_5}, \mathbf{C_6})$-8-foil decomposition of $\mathbf{K_{89}}$.
$B_i = \{(i, i+17, i+26, i+67, i+33), (i, i+1, i+6, i+22, i+44, i+21)\}$
$\cup \{(i, i+18, i+28, i+71, i+35), (i, i+2, i+8, i+23, i+51, i+27)\}$
$\cup \{(i, i+19, i+30, i+75, i+37), (i, i+3, i+10, i+24, i+54, i+29)\}$
$\cup \{(i, i+20, i+32, i+79, i+39), (i, i+4, i+12, i+25, i+57, i+31)\}$
$(i = 1, 2, ..., 89)$.

Example 5. A balanced $(\mathbf{C_5}, \mathbf{C_6})$-10-foil decomposition of $\mathbf{K_{111}}$.
$B_i = \{(i, i+21, i+32, i+83, i+41), (i, i+1, i+7, i+27, i+54, i+26)\}$
$\cup \{(i, i+22, i+34, i+87, i+43), (i, i+2, i+9, i+28, i+62, i+33)\}$
$\cup \{(i, i+23, i+36, i+91, i+45), (i, i+3, i+11, i+29, i+65, i+35)\}$
$\cup \{(i, i+24, i+38, i+95, i+47), (i, i+4, i+13, i+30, i+68, i+37)\}$
$\cup \{(i, i+25, i+40, i+99, i+49), (i, i+5, i+15, i+31, i+71, i+39)\}$
$(i = 1, 2, ..., 111)$.

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