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A Generalized Ryuoh-Nim:A Variant of the classical game of Wythoff Nim

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Abstract: We introduce the impartial game of *Ryūō* Nim, a variant of the classical game of Wythoff Nim. In the latter game, two players take turns in moving a single Queen of Chess on a large board, attempting to be the first to put her in the lower left corner, position (0,0). Instead of the queen used in Wythoff Nim, we use the *Ryūō* that is a promoted Hisha (rook) piece of Japanese chess. The *Ryūō* combines the power of the rook and king in western chess. We prove that the Grundy number for this variant is expressed by $\mathcal{G}((x, y)) = mod(x + y, 3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor)$, where $mod(x + y, 3)$ is the remaider of $x + y$ when divided by 3. We study a generalization of the *Ryūō* Nim whose Grudy number is expressed by $mod(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor)$ for a natural number p. We also study a generalized *Ryūō* Nim with a pass.

Keywords: Wythoff Nim Corner the Queen Problem Grundy Number nim-sum

1. *Ryūō* (dragon king)game

We introduce the impartial game of *Ryūō* Nim, a variant of the classical game of Wythoff Nim in [1]. Let $Z_{\geq 0}$ be the set of non-negative integers and N be the set of natural numbers. Instead of the queen used in Wythoff Nim, we use the *Ryūō* (dragon king) of Japanese chess. The *Ryūō* combines the power of the rook and king in western chess. The *Ryūō* is placed on a chess board of unbounded size, and two players move *Ryūō* in turns. The *Ryūō* can be moved horizontally, as far as one wants, he can be moved vertically, as far as one wants, and he can be moved one square diagonally.

Let us break with chess traditions here and name fields on the chess board by pairs of numbers. The field in the lower left corner will be called $(0, 0)$, and the other ones according to a Cartesian scheme - field (x, y) will be x fields to the right, and then y fields up (you get the picture, see Figure 2.

In our game we restrict the *Ryūō* to be moved to the left or upwards, or along the upper left diagonal, see Figure 1. Of course, the *Ryūō* has to be moved at least one field in each move.

The goal of the game is to move *Ryūō* to the "winning field" $(0, 0)$; whoever moves the *Ryūō* to this field, wins the game.

In this article we only treat impartical games. See [2] or [3] for a background on impartial games.

In this article we study impartial games without draws, so there will only be two outcome classes.

Definition 1.1. (a) $\mathcal{N}$-positions, from which the next player can force a win, as long as he plays correctly at every stage.

(b) $\mathcal{P}$-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.

Definition 1.2. For any position $\mathbf{p}$ of a game G, there is a set of positions that can be reached by making precisely one move in G, which we will denote by $move(\mathbf{p})$.

The move of *Ryūō* is expressed in (15), (16) and (17).

$$\begin{aligned} & move(x, y) \\ & = \{(u, y) : u < x\} && (1) \\ & \cup \{(x, v) : v < y\} && (2) \\ & \cup \{(x - 1, y - 1)\}. && (3) \end{aligned}$$

The set (15) stands for the horizontal move, the set (16) stands for the vertical move, and the set (17) stands for the diagonal move in Figure 2.

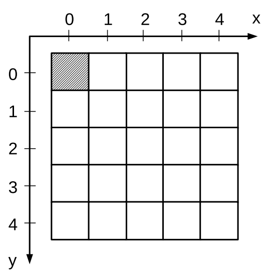

Fig. 1 definition of coordinates

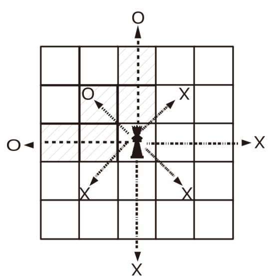

Fig. 2 move of *Ryūō*

Definition 1.3. (i) The *minimum excluded value* (mex) of a set, S, of non-negative integers is the least non-negative integer which is not in S.

(ii) Each position $\mathbf{p}$ of a impartial game G has an associated Grundy number, and we denoted it by $\mathcal{G}(\mathbf{p})$.

Grundy number is calculated recursively: $\mathcal{G}(\mathbf{p}) = mex\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

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Example 1.1. Examples of calculation of mex.

$$mex\{0,1,2,3\} = 4, mex\{1,1,2,3\} = 0,$$
$$mex\{0,2,3,5\} = 1 \text{ and } mex\{0,0,0,1\} = 2.$$

Theorem 1.1. Let $\mathcal{G}$ be the *Grundy* number. Then, $\mathbf{h}$ is a $\mathcal{P}$-*position* if and only if $\mathcal{G}(\mathbf{h}) = 0$.

This is a well known theorem in combinatorial game theory.

Example 1.2. Figure 3 is a table of Grundy numbers of the *Ryūō*-nim.

Y\X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	0	4	5	3	7	8	6	10	11	9	13
2	2	0	1	5	3	4	8	6	7	11	9	10	14
3	3	4	5	0	1	2	9	10	11	6	7	8	15
4	4	5	3	1	2	0	10	11	9	7	8	6	16
5	5	3	4	2	0	1	11	9	10	8	6	7	17
6	6	7	8	9	10	11	0	1	2	3	4	5	18
7	7	8	6	10	11	9	1	2	0	4	5	3	19
8	8	6	7	11	9	10	2	0	1	5	3	4	20
9	9	10	11	6	7	8	3	4	5	0	1	2	21
10	10	11	9	7	8	6	4	5	3	1	2	0	22
11	11	9	10	8	6	7	5	3	4	2	0	1	23
12	12	13	14	15	16	17	18	19	20	21	22	23	0

Fig. 3 the Grundy number $\mathcal{G}((x,y))$ of *Ryūō*

Y\X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	0	4	5	3	7	8	6	10	11	9	13
2	2	0	1	5	3	4	8	6	7	11	9	10	14
3	3	4	5	0	1	2	9	10	11	6	7	8	15
4	4	5	3	1	2	0	10	11	9	7	8	6	16
5	5	3	4	2	0	1	11	9	10	8	6	7	17
6	6	7	8	9	10	11	0	1	2	3	4	5	18
7	7	8	6	10	11	9	1	2	0	4	5	3	19
8	8	6	7	11	9	10	2	0	1	5	3	4	20
9	9	10	11	6	7	8	3	4	5	0	1	2	21
10	10	11	9	7	8	6	4	5	3	1	2	0	22
11	11	9	10	8	6	7	5	3	4	2	0	1	23
12	12	13	14	15	16	17	18	19	20	21	22	23	0

Fig. 4 $mod(x+y,3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor)$

It is easy to see that Figure 3 is the same as Figure 4. This implies that $\mathcal{G}((x,y)) = mod(x+y,3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor)$. This fact is presented in Theorem 1.2.

Next, we generalize *Ryūō*, and we define a generalized *Ryūō*.

Definition 1.4. Let p a natural number. We define a generalized *Ryūō* for p. For the generalized *Ryūō*, move is expressed in (4), (5) and (6).

$$\begin{aligned}
&move(x,y) \\
&= \{(u,y) : u < x\} && (4)\\
&\cup \{(x,v) : v < y\} && (5)\\
&\cup \{(x-s, y-t) : 1 \le s,t \text{ and } s+t \le p-1\}. && (6)
\end{aligned}$$

The set (4) stands for the horizontal move, the set (5) stands for the vertical move, and the set (6) stands for the upper left move.

Example 1.3. Figure 5 is the move of the *Ryūō*. Figure 6 and Figure 7 are the move of the generalized *Ryūō* for $p = 4$ and $p = 8$ respectively. Figure 8 represent the move of generalized *Ryūō* for a natural number p.

and other figures are moves of generalized *Ryūō*s.

In our game we restrict these to be moved to the left or upwards, or along the upper upper left. Of course, each has to be moved at least one field in each move.

In these figures, the sets (4) and (5) are expressed by dotted lines, and the set (6) is expressed as a set of small circles.

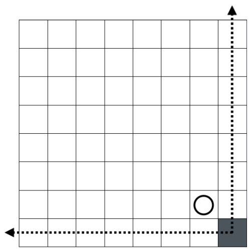

Fig. 5 move of the *Ryūō*

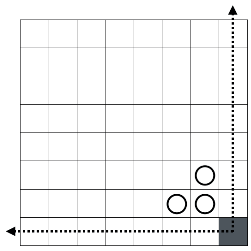

Fig. 6 move of a general *Ryūō* for $p = 4$.

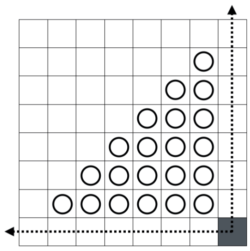

Fig. 7 move of a general *Ryūō* for $p = 8$.

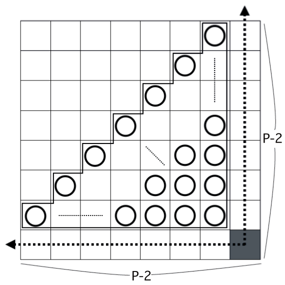

Fig. 8 move of a general *Ryūō* for p.

Example 1.4. Figure 9 is a table of Grundy numbers of the generalized *Ryūō*-nim for $p = 4$.

Y\X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	0	5	6	7	4	9	10	11	8	13
2	2	3	0	1	6	7	4	5	10	11	8	9	14
3	3	0	1	2	7	4	5	6	11	8	9	10	15
4	4	5	6	7	0	1	2	3	12	13	14	15	8
5	5	6	7	4	1	2	3	0	13	14	15	12	9
6	6	7	4	5	2	3	0	1	14	15	12	13	10
7	7	4	5	6	3	0	1	2	15	12	13	14	11
8	8	9	10	11	12	13	14	15	0	1	2	3	4
9	9	10	11	8	13	14	15	12	1	2	3	0	5
10	10	11	8	9	14	15	12	13	2	3	0	1	6
11	11	8	9	10	15	12	13	14	3	0	1	2	7
12	12	13	14	15	8	9	10	11	4	5	6	7	0

Fig. 9 the Grundy numbers of the generalized $\mathcal{G}((x,y))$ of *Ryūō* for $p = 4$

Y\X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	0	5	6	7	4	9	10	11	8	13
2	2	3	0	1	6	7	4	5	10	11	8	9	14
3	3	0	1	2	7	4	5	6	11	8	9	10	15
4	4	5	6	7	0	1	2	3	12	13	14	15	8
5	5	6	7	4	1	2	3	0	13	14	15	12	9
6	6	7	4	5	2	3	0	1	14	15	12	13	10
7	7	4	5	6	3	0	1	2	15	12	13	14	11
8	8	9	10	11	12	13	14	15	0	1	2	3	4
9	9	10	11	8	13	14	15	12	1	2	3	0	5
10	10	11	8	9	14	15	12	13	2	3	0	1	6
11	11	8	9	10	15	12	13	14	3	0	1	2	7
12	12	13	14	15	8	9	10	11	4	5	6	7	0

Fig. 10 $mod(x+y,4) + 4(\lfloor \frac{x}{4} \rfloor \oplus \lfloor \frac{y}{4} \rfloor)$

It is easy to see that Figure 9 is the same as Figure 10. This implies that $\mathcal{G}((x,y)) = mod(x+y,4) + 4(\lfloor \frac{x}{4} \rfloor \oplus \lfloor \frac{y}{4} \rfloor)$. In general the Grundy number of the generalized $\mathcal{G}((x,y))$ of *Ryūō* for p is $\mathcal{G}((x,y)) = mod(x+y,p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor)$. We present this fact in Theorem 1.2.

We present some lemmas that are needed for Theorem 1.2 without proof. We present Theorem 1.2 without proof, since the proof is too lengthy.

Lemma 1.1.

$$k \oplus h = mex(\{(k-t) \oplus h : t = 1,2,...,k\} \cup \{k \oplus (h-t) : t = 1,2,...,h\}). \quad (7)$$

Lemma 1.2. Let $A_{k,h} = \bigcup_{u=0}^{p-1}\{p((k-t)\oplus h) + u : t = 1,2,...,k\} \cup \bigcup_{u=0}^{p-1}\{p(k\oplus(h-t)) + u : t = 1,2,...,h\}$. (a) For any $v = 1,...,p-1$

$$p(k \oplus h) + v = mex(A_{k,h} \cup \{p(k\oplus h) + w : w = 0,..,v-1\}). \quad (8)$$

(b) For $v = 0$ $p(k \oplus h) = mex(A_{k,h})$.

For any arbitrary non-negative integer x, we denote by $mod(x, p)$ the remainder of x when divided by p.

Lemma 1.3. Let $x, y, k \in Z_{\geq 0}$. If $0 \leq k < mod(x+y, p)$, then

$$k \in \{mod(x-s+y-t, p) : 0 \leq s \leq x, 0 \leq t \leq y \text{ and } \leq s+t \leq p-1\}. \quad (9)$$

Lemma 1.4. Let V be a subset of $Z_{\geq 0}$, and let $v \in Z_{\geq 0}$ such that

$$v = mex(V). \quad (10)$$

If W is a subset of $Z_{\geq 0}$ such that $V \subset W$ and $v \notin W$, then $v = mex(W)$.

Lemma 1.5. Let $k, h, v, w \in Z_{\geq 0}$ such that $0 \leq v, w \leq p-1$, and let

$$\begin{aligned} C_{k,h,v,w} &= \{p(k\oplus h) + mod(v-t+w, p) : t = 1,2,...,v\}, \\ &\cup \{p(k\oplus h) + mod(v+w-t, p) : t = 1,2,...,w\}, \\ &\cup \{p(\lfloor \frac{pk+v-s}{p} \rfloor \oplus \lfloor \frac{ph+w-t}{p} \rfloor) + mod(v+w-s-t, p) \\ &: 1 \leq s, t \text{ and } s+t \leq p-1\}. \end{aligned} \quad (11)$$

Then we have the following (a) and (b).

(a) $p(k\oplus h) + mod(v+w, p) \notin C_{k,h,v,w}$

(b) $p(k\oplus h) + u \in C_{k,h,v,w}$ for any non-negative integer u such that

$$0 \leq u < mod(v+w, p). \quad (12)$$

Theorem 1.2.

$$\mathcal{G}((x,y)) = mod(x+y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor). \quad (13)$$

Here, $mod(x+y, p)$ is the remainder of $x+y$ when divided by p.

Theorem 1.2 and Theorem 1.3 present a sufficient condition and a necessary condition for a chess piece to have Grundy numbers expressed by (13) respectively.

Theorem 1.3. Suppose that we make a variant of "Corner the Queen" with a new chess piece. If the Grundy number of this game satisfies (14), then the move of this piece is defined by (4), (5) and (6) of Definition 1.4.

$$\mathcal{G}'((x,y)) = mod(x+y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor). \quad (14)$$

2. *Ryūō* Nim and a generalized *Ryūō* Nim with a pass

In this section the authors present the research on *Ryūō* Nim and a generalized *Ryūō* Nim with a pass.

Definition 2.1. For any position **p** of a game G, there is a set of positions that can be reached by making precisely one move in G, which we will denote by $move(\mathbf{p})$.

The move of *Ryūō* is expressed in (15), (16), (17), 18, 19, 20 and 21.

$$\begin{aligned} &move(x,y,0) \\ &= \{(u,y,0) : u < x\} \quad (15) \\ &\cup \{(x,v,0) : v < y\} \quad (16) \\ &\cup \{(x-1, y-1, 0)\}. \quad (17) \end{aligned}$$

$$\begin{aligned} &move(x,y,1) \\ &= \{(u,y,1) : u < x\} \quad (18) \\ &\cup \{(x,v,1) : v < y\} \quad (19) \\ &\cup \{(x-1, y-1, 1)\} \quad (20) \\ &\cup \{(x,y,0)\} \quad (21) \end{aligned}$$

The sets (15) and (18) stand for the horizontal move, the sets (16) and (19) stand for the vertical move, the sets (17) and (20) stand for the diagonal move in Figure 2, and the set (21) stands for a pass move.

Let $P_0 = \{(x,y,0) : x+y = 0(\ mod\ p) \text{ and } \lfloor \frac{x}{p} \rfloor = \lfloor \frac{y}{p} \rfloor\}$. Let $P_{1,1} = \{(1+m, p-m) : 0 \leq m \leq p-1 \text{ and } m \in Z_{\geq 0}\}$, $P_{1,2} = \{(1+pn, 1+pn) : n \in N\}$ and $P_{1,3} = \{(k+pn, p+2-k+pn) : n \in N, 2 \leq k \leq p \text{ and } k \in Z_{\geq 0}\}$. Let $P_1 = P_{1,1} \cup P_{1,2} \cup P_{1,3}$. Let $P = P_0 \cup P_1$.

Lemma 2.1. $(x,y,0)$ is a P-position if and only if $(x,y,0) \in P_0$.

Theorem 2.1. $G((x,y,1)) = 0$ if and only if $(x,y,1) \in P_1$.

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