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Balanced C_{18}-Trefoil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{18} be *the cycle* on 18 vertices. *The* C_{18}-*trefoil* is a graph of 3 edge-disjoint C_{18}'s with a common vertex and the common vertex is called *the center of the* C_{18}-*trefoil*. When K_n is decomposed into edge-disjoint sum of C_{18}-trefoils, it is called that K_n *has a* C_{18}-*trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_{18}-trefoils, it is called that K_n *has a balanced* C_{18}-*trefoil decomposition* and this number is called *the replication number*.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[7]. Horák and Rosa[2] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a* C_3-*bowtie system*.

In this sense, our balanced C_{18}-trefoil decomposition of K_n is to be known as *a balanced* C_{18}-*trefoil system*.

2. Balanced C_{18}-trefoil decomposition of K_n

Notation. We denote a C_{18}-trefoil passing through $v_1-v_2-v_3-v_4-v_5-v_6-v_7-v_8-v_9-v_{10}-v_{11}-v_{12}-v_{13}-v_{14}-v_{15}-v_{16}-v_{17}-v_{18}-v_1$, $v_1-v_{19}-v_{20}-v_{21}-v_{22}-v_{23}-v_{24}-v_{25}-v_{26}-v_{27}-v_{28}-v_{29}-v_{30}-v_{31}-v_{32}-v_{33}-v_{34}-v_{35}-v_1$, $v_1-v_{36}-v_{37}-v_{38}-v_{39}-v_{40}-v_{41}-v_{42}-v_{43}-v_{44}-v_{45}-v_{46}-v_{47}-v_{48}-v_{49}-v_{50}-v_{51}-v_{52}-v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}), (v_1, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}), (v_1, v_{36}, v_{37}, v_{38}, v_{39}, v_{40}, v_{41}, v_{42}, v_{43}, v_{44}, v_{45}, v_{46}, v_{47}, v_{48}, v_{49}, v_{50}, v_{51}, v_{52})\}$.

Theorem 1. K_n has a balanced C_{18}-trefoil decomposition if and only if $n \equiv 1$ (mod 108).

Proof. **(Necessity)** Suppose that K_n has a balanced C_{18}-trefoil decomposition. Let b be the number of C_{18}-trefoils and r be the replication number. Then $b = n(n-1)/108$ and $r = 52(n-1)/108$. Among r C_{18}-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{18}-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/108$ and $r_2 = 51(n-1)/108$. Therefore, $n \equiv 1$ (mod 108) is necessary.

(Sufficiency) Put $n = 108t+1$. Construct tn C_{18}-trefoils as follows:

$B_i^{(1)} = \{\ (i, i+1, i+6t+2, i+36t+2, i+54t+3, i+72t+3, i+6t+3, i+48t+3, i+78t+4, i+27t+3, i+84t+4, i+51t+3, i+12t+3, i+75t+3, i+60t+3, i+39t+2, i+12t+2, i+3t+1),$
$(i, i+2, i+6t+4, i+36t+3, i+54t+5, i+72t+4, i+6t+5, i+48t+4, i+78t+6, i+27t+4, i+84t+6, i+51t+4, i+12t+5, i+75t+4, i+60t+5, i+39t+3, i+12t+4, i+3t+2),$
$(i, i+3, i+6t+6, i+36t+4, i+54t+7, i+72t+5, i+6t+7, i+48t+5, i+78t+8, i+27t+5, i+84t+8, i+51t+5, i+12t+7, i+75t+5, i+60t+7, i+39t+4, i+12t+6, i+3t+3)\ \}$

$B_i^{(2)} = \{\ (i, i+4, i+6t+8, i+36t+5, i+54t+9, i+72t+6, i+6t+9, i+48t+6, i+78t+10, i+27t+6, i+84t+10, i+51t+6, i+12t+9, i+75t+6, i+60t+9, i+39t+5, i+12t+8, i+3t+4),$
$(i, i+5, i+6t+10, i+36t+6, i+54t+11, i+72t+7, i+6t+11, i+48t+7, i+78t+12, i+27t+7, i+84t+12, i+51t+7, i+12t+11, i+75t+7, i+60t+11, i+39t+6, i+12t+10, i+3t+5),$
$(i, i+6, i+6t+12, i+36t+7, i+54t+13, i+72t+8, i+6t+13, i+48t+8, i+78t+14, i+27t+8, i+84t+14, i+51t+8, i+12t+13, i+75t+8, i+60t+13, i+39t+7, i+12t+12, i+3t+6)\}$

...

$B_i^{(t)} = \{\ (i, i+3t-2, i+12t-4, i+39t-1, i+60t-$

$3, i+75t, i+12t-3, i+51t, i+84t-2, i+30t, i+90t-2, i+54t, i+18t-3, i+78t, i+66t-3, i+42t-1, i+18t-4, i+6t-2)$,
$(i, i+3t-1, i+12t-2, i+39t, i+60t-1, i+75t+1, i+12t-1, i+51t+1, i+84t, i+30t+1, i+90t, i+54t+1, i+18t-1, i+78t+1, i+66t-1, i+42t, i+18t-2, i+6t-1)$,
$(i, i+3t, i+12t, i+39t+1, i+60t+1, i+75t+2, i+12t+1, i+51t+2, i+84t+2, i+30t+2, i+90t+2, i+54t+2, i+18t+1, i+78t+2, i+66t+1, i+42t+1, i+18t, i+6t)\ \}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_{18}-trefoil decomposition of K_n.

Example 1. Balanced C_{18}-trefoil decomposition of K_{109}.
$B_i = \{(i, i+1, i+8, i+38, i+57, i+75, i+9, i+51, i+82, i+30, i+88, i+54, i+15, i+78, i+63, i+41, i+14, i+4)$,
$(i, i+2, i+10, i+39, i+59, i+76, i+11, i+52, i+84, i+31, i+90, i+55, i+17, i+79, i+65, i+42, i+16, i+5)$,
$(i, i+3, i+12, i+40, i+61, i+77, i+13, i+53, i+86, i+32, i+92, i+56, i+19, i+80, i+67, i+43, i+18, i+6)\}$
$(i = 1, 2, ..., 109)$.

Example 2. Balanced C_{18}-trefoil decomposition of K_{217}.
$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+15, i+99, i+160, i+57, i+172, i+105, i+27, i+153, i+123, i+80, i+26, i+7)$,
$(i, i+2, i+16, i+75, i+113, i+148, i+17, i+100, i+162, i+58, i+174, i+106, i+29, i+154, i+125, i+81, i+28, i+8)$,
$(i, i+3, i+18, i+76, i+115, i+149, i+19, i+101, i+164, i+59, i+176, i+107, i+31, i+155, i+127, i+82, i+30, i+9)\}$
$B_i^{(2)} = \{(i, i+4, i+20, i+77, i+117, i+150, i+21, i+102, i+166, i+60, i+178, i+108, i+33, i+156, i+129, i+83, i+32, i+10)$,
$(i, i+5, i+22, i+78, i+119, i+151, i+23, i+103, i+168, i+61, i+180, i+109, i+35, i+157, i+131, i+84, i+34, i+11)$,
$(i, i+6, i+24, i+79, i+121, i+152, i+25, i+104, i+170, i+62, i+182, i+110, i+37, i+158, i+133, i+85, i+36, i+12)\}\quad (i = 1, 2, ..., 217)$.

Example 3. Balanced C_{18}-trefoil decomposition of K_{325}.
$B_i^{(1)} = \{(i, i+1, i+20, i+110, i+165, i+219, i+21, i+147, i+238, i+84, i+256, i+156, i+39, i+228, i+183, i+119, i+38, i+10)$,
$(i, i+2, i+22, i+111, i+167, i+220, i+23, i+148, i+240, i+85, i+258, i+157, i+41, i+229, i+185, i+120, i+40, i+11)$,
$(i, i+3, i+24, i+112, i+169, i+221, i+25, i+149, i+242, i+86, i+260, i+158, i+43, i+230, i+187, i+121, i+42, i+12)\}$
$B_i^{(2)} = \{(i, i+4, i+26, i+113, i+171, i+222, i+27, i+150, i+244, i+87, i+262, i+159, i+45, i+231, i+189, i+122, i+44, i+13)$,
$(i, i+5, i+28, i+114, i+173, i+223, i+29, i+151, i+246, i+88, i+264, i+160, i+47, i+232, i+191, i+123, i+46, i+14)$,
$(i, i+6, i+30, i+115, i+175, i+224, i+31, i+152, i+248, i+89, i+266, i+161, i+49, i+233, i+193, i+124, i+48, i+15)\}$
$B_i^{(3)} = \{(i, i+7, i+32, i+116, i+177, i+225, i+33, i+153, i+250, i+90, i+268, i+162, i+51, i+234, i+195, i+125, i+50, i+16)$,
$(i, i+8, i+34, i+117, i+179, i+226, i+35, i+154, i+252, i+91, i+270, i+163, i+53, i+235, i+197, i+126, i+52, i+17)$,
$(i, i+9, i+36, i+118, i+181, i+227, i+37, i+155, i+254, i+92, i+272, i+164, i+55, i+236, i+199, i+127, i+54, i+18)\}\quad (i = 1, 2, ..., 325)$.

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