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# Balanced $C_{18}$ -Bowtie Decomposition Algorithm of Complete Graphs

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## 1. Introduction

Let  $K_n$  denote the complete graph of  $n$  vertices. Let  $C_{18}$  be the cycle on 18 vertices. The  $C_{18}$ -bowtie is a graph of 2 edge-disjoint  $C_{18}$ 's with a common vertex and the common vertex is called the center of the  $C_{18}$ -bowtie. When  $K_n$  is decomposed into edge-disjoint sum of  $C_{18}$ -bowties, it is called that  $K_n$  has a  $C_{18}$ -bowtie decomposition. Moreover, when every vertex of  $K_n$  appears in the same number of  $C_{18}$ -bowties, it is called that  $K_n$  has a balanced  $C_{18}$ -bowtie decomposition and this number is called the replication number.

It is a well-known result that  $K_n$  has a  $C_3$  decomposition if and only if  $n \equiv 1$  or  $3 \pmod{6}$ . This decomposition is known as a Steiner triple system. See Colbourn and Rosa[1] and Wallis[7]. Horák and Rosa[2] proved that  $K_n$  has a  $C_3$ -bowtie decomposition if and only if  $n \equiv 1$  or  $9 \pmod{12}$ . This decomposition is known as a  $C_3$ -bowtie system.

In this sense, our balanced  $C_{18}$ -bowtie decomposition of  $K_n$  is to be known as a balanced  $C_{18}$ -bowtie system.

## 2. Balanced $C_{18}$ -bowtie decomposition of $K_n$

**Notation.** We denote a  $C_{18}$ -bowtie passing through  $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_1$ ,  $v_1 - v_{19} - v_{20} - v_{21} - v_{22} - v_{23} - v_{24} - v_{25} - v_{26} - v_{27} - v_{28} - v_{29} - v_{30} - v_{31} - v_{32} - v_{33} - v_{34} - v_{35} - v_1$  by  $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}), (v_1, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35})\}$ .

**Theorem.**  $K_n$  has a balanced  $C_{18}$ -bowtie decomposition if and only if  $n \equiv 1 \pmod{72}$ .

**Proof. (Necessity)** Suppose that  $K_n$  has a balanced  $C_{18}$ -bowtie decomposition. Let  $b$  be the number of  $C_{18}$ -bowties and  $r$  be the replication number. Then  $b = n(n-1)/72$  and  $r = 35(n-1)/72$ . Among  $r$   $C_{18}$ -bowties having a vertex  $v$  of  $K_n$ , let  $r_1$  and  $r_2$  be the numbers of  $C_{18}$ -bowties in which  $v$  is the center and  $v$  is not the center, respectively. Then  $r_1 + r_2 = r$ . Counting the number of vertices adjacent to  $v$ ,  $4r_1 + 2r_2 = n-1$ . From these relations,  $r_1 = (n-1)/72$  and  $r_2 = 34(n-1)/72$ . Therefore,  $n \equiv 1 \pmod{72}$  is necessary.

**(Sufficiency)**  $n = 72t+1$ . Construct  $t n$   $C_{18}$ -bowties as follows:

$$B_i^{(1)} = \{ (i, i+1, i+4t+2, i+24t+2, i+36t+3, i+48t+3, i+4t+3, i+32t+3, 52t+4, i+18t+3, 56t+4, i+34t+3, i+8t+3, i+50t+3, i+40t+3, i+26t+2, i+8t+2, i+2t+1),$$

$$(i, i+2, i+4t+4, i+24t+3, i+36t+5, i+48t+4, i+4t+5, i+32t+4, 52t+6, i+18t+4, 56t+6, i+34t+4, i+8t+5, i+50t+4, i+40t+5, i+26t+3, i+8t+4, i+2t+2) \}$$

$$B_i^{(2)} = \{ (i, i+3, i+4t+6, i+24t+4, i+36t+7, i+48t+5, i+4t+7, i+32t+5, 52t+8, i+18t+5, 56t+8, i+34t+5, i+8t+7, i+50t+5, i+40t+7, i+26t+4, i+8t+6, i+2t+3),$$

$$(i, i+4, i+4t+8, i+24t+5, i+36t+9, i+48t+6, i+4t+9, i+32t+6, 52t+10, i+18t+6, 56t+10, i+34t+6, i+8t+9, i+50t+6, i+40t+9, i+26t+5, i+8t+8, i+2t+4) \}$$

$$\dots$$

$$B_i^{(t)} = \{ (i, i+2t-1, i+8t-2, i+26t, i+40t-1, i+50t+1, i+8t-1, i+34t+1, 56t, i+20t+1, 60t, i+36t+1, i+12t-1, i+52t+1, i+44t-1, i+28t, i+12t-2, i+4t-1), (i, i+2t, i+8t, i+26t+1, i+40t+1, i+50t+2, i+8t+1, i+34t+2, 56t+2, i+20t+2, 60t+2, i+36t+2, i+12t+1, i+52t+2, i+44t+1, i+28t+1, i+12t, i+4t) \}$$

$$(i=1, 2, \dots, n).$$

Then they comprise a balanced  $C_{18}$ -bowtie de-

composition of  $K_n$ .

**Example 1. Balanced  $C_{18}$ -bowtie decomposition of  $K_{73}$ .**

$$B_i = \{(i, i+1, i+6, i+26, i+39, i+51, i+7, i+35, i+56, i+21, i+60, i+37, i+11, i+53, i+43, i+28, i+10, i+3), \\ (i, i+2, i+8, i+27, i+41, i+52, i+9, i+36, i+58, i+22, i+62, i+38, i+13, i+54, i+45, i+29, i+12, i+4)\} \quad (i=1, 2, \dots, 73).$$

**Example 2. Balanced  $C_{18}$ -bowtie decomposition of  $K_{145}$ .**

$$B_i^{(1)} = \{(i, i+1, i+10, i+50, i+75, i+99, i+11, i+67, i+108, i+39, i+116, i+71, i+19, i+103, i+83, i+54, i+18, i+5), \\ (i, i+2, i+12, i+51, i+77, i+100, i+13, i+68, i+110, i+40, i+118, i+72, i+21, i+104, i+85, i+55, i+20, i+6)\} \\ B_i^{(2)} = \{(i, i+3, i+14, i+52, i+79, i+101, i+15, i+69, i+112, i+41, i+120, i+73, i+23, i+105, i+87, i+56, i+22, i+7), \\ (i, i+4, i+16, i+53, i+81, i+102, i+17, i+70, i+114, i+42, i+122, i+74, i+25, i+106, i+89, i+57, i+24, i+8)\} \quad (i=1, 2, \dots, 145).$$

**Example 3. Balanced  $C_{18}$ -bowtie decomposition of  $K_{217}$ .**

$$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+15, i+99, i+160, i+57, i+172, i+105, i+27, i+153, i+123, i+80, i+26, i+7), \\ (i, i+2, i+16, i+75, i+113, i+148, i+17, i+100, i+162, i+58, i+174, i+106, i+29, i+154, i+125, i+81, i+28, i+8)\} \\ B_i^{(2)} = \{(i, i+3, i+18, i+76, i+115, i+149, i+19, i+101, i+164, i+59, i+176, i+107, i+31, i+155, i+127, i+82, i+30, i+9), \\ (i, i+4, i+20, i+77, i+117, i+150, i+21, i+102, i+166, i+60, i+178, i+108, i+33, i+156, i+129, i+83, i+32, i+10)\} \\ B_i^{(3)} = \{(i, i+5, i+22, i+78, i+119, i+151, i+23, i+103, i+168, i+61, i+180, i+109, i+35, i+157, i+131, i+84, i+34, i+11), \\ (i, i+6, i+24, i+79, i+121, i+152, i+25, i+104, i+170, i+62, i+182, i+110, i+37, i+158, i+133, i+85, i+36, i+12)\} \quad (i=1, 2, \dots, 217).$$

**Example 4. Balanced  $C_{18}$ -bowtie decomposition of  $K_{289}$ .**

$$B_i^{(1)} = \{(i, i+1, i+18, i+98, i+147, i+195, i+$$

$$19, i+131, i+212, i+75, i+228, i+139, i+35, i+203, i+163, i+106, i+34, i+9), \\ (i, i+2, i+20, i+99, i+149, i+196, i+21, i+132, i+214, i+76, i+230, i+140, i+37, i+204, i+165, i+107, i+36, i+10)\} \\ B_i^{(2)} = \{(i, i+3, i+22, i+100, i+151, i+197, i+23, i+133, i+216, i+77, i+232, i+141, i+39, i+205, i+167, i+108, i+38, i+11), \\ (i, i+4, i+24, i+101, i+153, i+198, i+25, i+134, i+218, i+78, i+234, i+142, i+41, i+206, i+169, i+109, i+40, i+12)\} \\ B_i^{(3)} = \{(i, i+5, i+26, i+102, i+155, i+199, i+27, i+135, i+220, i+79, i+236, i+143, i+43, i+207, i+171, i+110, i+42, i+13), \\ (i, i+6, i+28, i+103, i+157, i+200, i+29, i+136, i+222, i+80, i+238, i+144, i+45, i+208, i+173, i+111, i+44, i+14)\} \\ B_i^{(4)} = \{(i, i+7, i+30, i+104, i+159, i+201, i+31, i+137, i+224, i+81, i+240, i+145, i+47, i+209, i+175, i+112, i+46, i+15), \\ (i, i+8, i+32, i+105, i+161, i+202, i+33, i+138, i+226, i+82, i+242, i+146, i+49, i+210, i+177, i+113, i+48, i+16)\} \quad (i=1, 2, \dots, 289).$$

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