1Q- 1 Balanced bowtie decomposition algorithm of symmetric complete tripartite multi-digraphs

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1. Introduction

Let $K^*_{n_1,n_2,n_3}$ denote *the symmetric complete tripartite digraph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each. *The symmetric complete tripartite multi-digraph* $\lambda K^*_{n_1,n_2,n_3}$ is the symmetric complete tripartite digraph $K^*_{n_1,n_2,n_3}$ in which every edge is taken λ times. *The bowtie* (or *the 2-windmill*) is a graph of 2 edge-disjoint triangles with a common vertex and the common vertex is called *the center of the bowtie.* When $\lambda K^*_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of bowties, it is called that $\lambda K^*_{n_1,n_2,n_3}$ *has a bowtie decomposition.* Moreover, when every vertex of $\lambda K^*_{n_1,n_2,n_3}$ appears in the same number of bowties, it is called that $\lambda K^*_{n_1,n_2,n_3}$ *has a balanced bowtie decomposition* and this number is called *the replication number.*

2. Balanced bowtie decomposition of $\lambda K^*_{n_1,n_2,n_3}$

Notation. We denote a bowtie passing through $v_1 - v_2 - v_3 - v_1 - v_4 - v_5 - v_1$ by $\{(v_1,v_2,v_3),(v_1,v_4,v_5)\}$.

Lemma 1. If $\lambda K^*_{n,n,n}$ has a balanced bowtie decomposition, then $s\lambda K^*_{n,n,n}$ has a balanced bowtie decomposition.

Lemma 2. If $\lambda K^*_{n,n,n}$ has a balanced bowtie decomposition, then $\lambda K^*_{sn,sn,sn}$ has a balanced bowtie decomposition.

Theorem 3. $\lambda K^*_{n_1,n_2,n_3}$ has a balanced bowtie decomposition if and only if
(i) $n_1 = n_2 = n_3 \equiv 0 \pmod 3$ for $\lambda \equiv 1,2 \pmod 3$ and
(ii) $n_1 = n_2 = n_3 \geq 2$ for $\lambda \equiv 0 \pmod 3$.

Proof. (Necessity) Suppose that $\lambda K^*_{n_1,n_2,n_3}$ has a balanced bowtie decomposition. Let b be the number of bowties and r be the replication number. Then $b = \lambda(n_1n_2+n_1n_3+n_2n_3)/3$ and $r = 5\lambda(n_1n_2+n_1n_3+n_2n_3)/3(n_1+n_2+n_3)$. Among r bowties having vertex v in V_i, ler r_{ij} be the number of bowties in which the centers are in V_j. Then $r_{11}+r_{12}+r_{13} = r_{21}+r_{22}+r_{23} = r_{31}+r_{32}+r_{33} = r$. Counting the number of vertices adjacent to vertex v in V_1, $2r_{11}+r_{12}+r_{13} = 2\lambda n_2$ and $2r_{11}+r_{12}+r_{13} = 2\lambda n_3$. Counting the number of vertices adjacent to vertex v in V_2, $r_{21}+2r_{22}+r_{23} = 2\lambda n_1$ and $r_{21}+2r_{22}+r_{23} = 2\lambda n_3$. Counting the number of vertices adjacent to vertex v in V_3, $r_{31}+r_{32}+2r_{33} = 2\lambda n_1$ and $r_{31}+r_{32}+2r_{33} = 2\lambda n_2$.
Therefore, $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = n$. Then $b = \lambda n^2$, $r = 5\lambda n/3$, $r_{11} = r_{22} = r_{33} = \lambda n/3$ and $r_{12}+r_{13} = r_{21}+r_{23} = r_{31}+r_{32} = 4\lambda n/3$. Thus $\lambda n \equiv 0 \pmod 3$. Since a bowtie is a subgraph of $\lambda K^*_{n,n,n}$, $n \geq 2$.
Therefore, (i) $n_1 = n_2 = n_3 \equiv 0 \pmod 3$ for $\lambda \equiv 1,2 \pmod 3$ and (ii) $n_1 = n_2 = n_3 \geq 2$ for $\lambda \equiv 0 \pmod 3$ are necessary.
(Sufficiency) Case (i). $n \equiv 0 \pmod 3$. Put $n = 3s$. When $s = 1$, let $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6\}$, $V_3 = \{7,8,9\}$.
Construct a balanced bowtie decomposition of $K^*_{3,3,3}$:
$B_1 = \{(1,5,8),(1,9,6)\}$
$B_2 = \{(2,6,9),(2,7,4)\}$
$B_3 = \{(3,4,7),(3,8,5)\}$
$B_4 = \{(4,8,2),(4,3,9)\}$
$B_5 = \{(5,9,3),(5,1,7)\}$
$B_6 = \{(6,7,1),(6,2,8)\}$
$B_7 = \{(7,2,5),(7,6,3)\}$
$B_8 = \{(8,3,6),(8,4,1)\}$
$B_9 = \{(9,1,4),(9,5,2)\}$.
Therefore, $\lambda K^*_{n,n,n}$ has a balanced bowtie decomposition.

Case (ii). $n \geq 2$ **and** $\lambda \equiv 0 \pmod 3$**.**
Let $V_1 = \{1, 2, ..., n\}$, $V_2 = \{1', 2', ..., n'\}$, $V_3 = \{1'', 2'', ..., n''\}$.
Construct a balanced bowtie decomposition of $3K^*_{n,n,n}$:
$B^{(1)}_{ij} = \{(i, j', (i+j-1)''), (i, (i+j)'', (j+1)')\}$
$B^{(2)}_{ij} = \{(i', j'', i+j-1), (i', i+j, (j+1)'')\}$
$B^{(3)}_{ij} = \{(i'', j, (i+j-1)'), (i'', (i+j)', j+1)\}$.
Therefore, $\lambda K^*_{n,n,n}$ has a balanced bowtie decomposition.

References

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[2] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).