4F-07 Bowtie-decomposition algorithm and trefoil-decomposition algorithm of complete tripartite multi-graphs

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1. Introduction

Let K_{n_1,n_2,n_3} denote *the complete tripartite graph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each. *The complete tripartite multi-graph* $\lambda K_{n_1,n_2,n_3}$ is the complete tripartite graph K_{n_1,n_2,n_3} in which every arc is taken λ times. Two edge-disjoint triangles with a common vertex is called *the bowtie* and the common vertex of the bowtie is called *the center of the bowtie.* When $\lambda K_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of bowties, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a bowtie-decomposition.*
Three edge-disjoint triangles with a common vertex is called *the trefoil* and the common vertex of the trefoil is called *the center of the trefoil.* When $\lambda K_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of trefoils, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a trefoil-decomposition.*

2. Bowtie-decomposition algorithm of $\lambda K_{n_1,n_2,n_3}$

Notation. For a bowtie passing $v_1 - v_2 - v_3 - v_1 - v_4 - v_5 - v_1$, we denote $(v_1; v_2, v_3; v_4, v_5)$.

Theorem 1. $\lambda K_{n_1,n_2,n_3}$ has a bowtie-decomposition if and only if $n_1 = n_2 = n_3 \equiv 0 \pmod 2$ for odd λ and $n_1 = n_2 = n_3 \geq 2$ for even λ.

Proof.(Necessity) Suppose that $\lambda K_{n_1,n_2,n_3}$ has a bowtie-decomposition. Let b be the number of bowties. Then $b = \lambda(n_1n_2 + n_1n_3 + n_2n_3)/6$. Among b bowties, let b_i be the number of bowties whose center-vertices are in V_i. Then $b = b_1 + b_2 + b_3$. Counting the number of edges between V_1 and V_2, we have $2b_1+2b_2+2b_3 = \lambda n_1n_2$. Counting the number of edges between V_1 and V_3, we have $2b_1+2b_2+2b_3 = \lambda n_1n_3$. Counting the number of edges between V_2 and V_3, we have $2b_1 + 2b_2 + 2b_3 = \lambda n_2n_3$. Then $\lambda n_1n_2 = \lambda n_1n_3 = \lambda n_2n_3$. Therefore, $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = n$. Then $b = \lambda n^2/2$. Therefore, we must have $\lambda n \equiv 0 \pmod 2$. Since a bowtie is a subgraph of $\lambda K_{n,n,n}$, we must have $n \geq 2$. Therefore, $n \equiv 0 \pmod 2$ for odd λ and $n \geq 2$ for even λ.
(Sufficiency) Put $n_1 = n_2 = n_3 = n$. Let $V_1 = \{1, 2, ..., n\}$, $V_2 = \{1', 2', ..., n'\}$, $V_3 = \{1'', 2'', ..., n''\}$.
Case 1. $n \equiv 0 \pmod 2$ and λ is odd. Construct $n^2/2$ bowties BT_{ij} $(i = 1, 2, ..., n\ ;\ j = 1, 2, ..., n/2)$ as following :
$BT_{ij} = (\ i\ ;\ (2j-1)'\ ,\ (i+2j-2)''\ ;\ (2j)'\ ,\ (i+2j-1)''\ )$,
where the additions are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a bowtie-decomposition of $K_{n,n,n}$. Construct this bowtie-decomposition repeatly λ times. Then we have a bowtie-decomposition of $\lambda K_{n,n,n}$.
Case 2. $n \geq 2$ and λ is even. Construct n^2 bowties BT_{ij} $(i = 1, 2, ..., n\ ;\ j = 1, 2, ..., n)$ as following :
$BT_{ij} = (\ i\ ;\ j'\ ,\ (i+j-1)''\ ;\ (j+1)'\ ,\ (i+j)''\ )$,
where the additions are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a bowtie-decomposition of $2K_{n,n,n}$. Construct this bowtie-decomposition repeatly $\lambda/2$ times. Then we have a bowtie-decomposition of $\lambda K_{n,n,n}$.

3. Trefoil-decomposition algorithm of $\lambda K_{n_1,n_2,n_3}$

Notation. For a trefoil passing $v_1 - v_2 - v_3 - v_1 - v_4 - v_5 - v_1 - v_6 - v_7 - v_1$, we denote $(v_1; v_2, v_3; v_4, v_5; v_6, v_7)$.

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E-mail:ushio@is.kindai.ac.jp 完全３組多重グラフの bowtie 分解アルゴリズムと trefoil 分解アルゴリズム 近畿大学理工学部経営工学科 潮 和彦

Theorem 2. $\lambda K_{n_1,n_2,n_3}$ has a trefoil-decomposition if and only if $n_1 = n_2 = n_3 \equiv 0 \pmod 3$ for $\lambda \equiv 1, 2 \pmod 3$ and $n_1 = n_2 = n_3 \geq 3$ for $\lambda \equiv 0 \pmod 3$.

Proof.(Necessity) Suppose that $\lambda K_{n_1,n_2,n_3}$ has a trefoil-decomposition. Let b be the number of trefoils. Then $b = \lambda(n_1n_2 + n_1n_3 + n_2n_3)/9$. Among b trefoils, let b_i be the number of trefoils whose center-vertices are in V_i. Then $b = b_1 + b_2 + b_3$. Counting the number of edges between V_1 and V_2, we have $3b_1+3b_2+3b_3 = \lambda n_1 n_2$. Counting the number of edges between V_1 and V_3, we have $3b_1+3b_2+3b_3 = \lambda n_1 n_3$. Counting the number of edges between V_2 and V_3, we have $3b_1 + 3b_2 + 3b_3 = \lambda n_2 n_3$. Then $\lambda n_1 n_2 = \lambda n_1 n_3 = \lambda n_2 n_3$. Therefore, $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = n$. Then $b = \lambda n^2/3$. Therefore, we must have $\lambda n \equiv 0 \pmod 3$. Since a trefoil is a subgraph of $\lambda K_{n,n,n}$, we must have $n \geq 3$. Therefore, $n \equiv 0 \pmod 3$ for $\lambda \equiv 1, 2 \pmod 3$ and $n \geq 3$ for $\lambda \equiv 0 \pmod 3$.
(Sufficiency) Put $n_1 = n_2 = n_3 = n$. Let $V_1 = \{1, 2, ..., n\}$, $V_2 = \{1', 2', ..., n'\}$, $V_3 = \{1'', 2'', ..., n''\}$.
Case 1. $n \equiv 0 \pmod 3$ and $\lambda \equiv 1, 2 \pmod 3$. Construct $n^2/3$ trefoils TF_{ij} ($i = 1, 2, ..., n$; $j = 1, 2, ..., n/3$) as following :
$TF_{ij} = (\ i \ ; (3j-2)'\ ,\ (i+3j-3)''\ ;\ (3j-1)'\ ,\ (i+3j-2)''\ ;\ (3j)'\ ,\ (i+3j-1)''\ )$,
where the additions are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a trefoil-decomposition of $K_{n,n,n}$. Construct this trefoil-decomposition repeatly λ times. Then we have a trefoil-decomposition of $\lambda K_{n,n,n}$.
Case 2. $n \geq 3$ and $\lambda \equiv 0 \pmod 3$. Construct n^2 trefoils TF_{ij} ($i = 1, 2, ..., n$; $j = 1, 2, ..., n$) as following :
$TF_{ij} = (\ i \ ;\ j'\ ,\ (i+j-1)''\ ;\ (j+1)'\ ,\ (i+j)''\ ;\ (j+2)'\ ,\ (i+j+1)''\ )$,
where the additions are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a trefoil-decomposition of $3K_{n,n,n}$. Construct this trefoil-decomposition repeatly $\lambda/3$ times. Then we have a trefoil-decomposition of $\lambda K_{n,n,n}$.

References

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