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The Simplest and Smallest Network on Which the Ford-Fulkerson Maximum Flow Procedure May Fail to Terminate

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Abstract: Ford and Fulkerson's labeling method is a classic algorithm for maximum network flows. The labeling method always terminates for networks whose edge capacities are integral (or, equivalently, rational). On the other hand, it might fail to terminate if networks have an edge with an irrational capacity. Ford and Fulkerson also gave an example of such networks on which the labeling method might fail to terminate. However, their example has 10 vertices and 48 edges and the flow augmentation is a little bit complicated. Simpler examples have been published in the past. In 1995, Zwick gave two networks with 6 vertices and 9 edges and one network with 6 vertices and 8 edges. The latter is the smallest, however, the calculation of the irrational capacity requires some effort. Thus, he called the former the *simplest*. In this paper, we show the simplest and smallest network in Zwick's context. Moreover, the irrational edge capacity of our example can be arbitrarily assigned while those in the all previous examples are not. This suggests that many real-valued networks might fail to terminate.

Keywords: maximum flow, Ford-Fulkerson algorithm, flow augmenting path, infinite continued fraction

1. Introduction

Ford and Fulkerson's labeling method is a classic maximum flow algorithm [1]. It repeats flow augmenting steps while the current residual network has a flow augmenting path from the source to the sink. Thus, the labeling method is also called the augmenting path method [9]. If all the edge capacities of a network are integral (or, equivalently, rational), the labeling method always terminates in finite steps. On the other hand, if a network has an edge with an irrational capacity, it might fail to terminate. Ford and Fulkerson also gave an example of such networks in Ref. [1].

Many textbooks describe the labeling method, however, few show Ford and Fulkerson's example [5], [6], [7], [10], [12]. One of the reasons is that it has 10 vertices and 48 edges and the flow augmenting step is complicated as described in Refs. [8], [13]. We refer to Ford and Fulkerson's example as N_0 and show it in **Fig. 1**, where an undirected edge (u, v) represents a pair of directed edges (u, v) and (v, u) for simplicity.

After publishing of the labeling method by Ford and Fulkerson, some techniques were developed so that the algorithm always terminates. (See Edmonds and Karp [4] and Dinic [2] for example.) Moreover, better algorithms, such as the preflow-push algorithm [11], have been devised. So people might regard the non-termination of the labeling method as insignificant. From educational point of view, however, it is worthwhile to study such a property since the labeling method is the basis of the max-flow min-cut theorem, the integrality theorem, and other maximum flow algorithms.

Chvátal [8], Korte and Vygen [14], and Bang-Jensen and Gutin [15] show simpler networks in their textbook. Above all, Zwick gave decisive examples: two networks N_1 (**Fig. 2**) and N_2 (**Fig. 3**) are the *simplest* and network N_3 (**Fig. 4**) is the smallest [13]. N_3 has six vertices and eight edges. Zwick wrote that the labeling method always terminates for any network with five or less vertices or with seven or less edges and called N_3 the smallest example. As described later, N_3 has two edges with irrational capacities determined by complicated procedures. The value of the flow converges to the maximum flow. N_1 and N_2 have six vertices and nine edges, however, they have only one edge with irrational capacity. Zwick called N_1 and N_2 the simplest examples. None of the flow values converges to the maximum flow.

In this paper, we modify Zwick's N_3 and obtain the simplest and smallest example. Moreover, the irrational edge capacity in this example can be arbitrarily assigned while those in all previous examples are not. This suggests that many networks with real-valued capacities have infinite sequences of flow augmentations.

2. Ford and Fulkerson's Example

Ford and Fulkerson gave the following network in Ref. [1]:

- Vertex set: $\{s, t, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$.
- Edge set: four *special edges* $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $A_3 = (x_3, y_3)$, $A_4 = (x_4, y_4)$, and (y_i, y_j), (x_i, y_j), (y_i, x_j), (s, x_i), and (y_i, t) for $i \neq j$.

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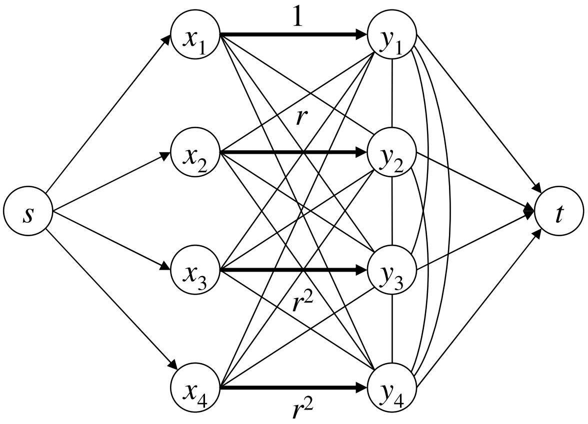


Fig. 1 Ford and Fulkerson's example N_0.

- Edge capacities: The capacities of A_1, A_2, A_3, and A_4 are r^0, r^1,r^2, and r^2, respectively, where $r = (\sqrt{5} - 1)/2 < 1$. The capacities of the other edges are $S = \sum_{n=0}^{\infty} r^n = (\sqrt{5} + 3)/2$.

Figure 1 shows the network, where an undirected edge (u, v) represents two directed edges (u, v) and (v, u). We call this example N_0.

Ford and Fulkerson showed that N_0 has an infinite sequence of flow augmentations which simulates the computation of series $\{a_n\}$ given by recurrence $a_{n+2} = a_n - a_{n+1}$. For initial condition $a_0 = 1$ and $a_1 = r$, $a_n = r^n$ is a solution of the recurrence, where $r = (\sqrt{5} - 1)/2 < 1$. In Procedure 1, the labeling method chooses flow augmenting paths so that the value of flows increases by r^n in every flow augmenting step.

Procedure 1 (Ford and Fulkerson [1])

[Initial flow] Choose a flow augmenting path p so that the special edge contained in p is only A_1. For example, $p = \langle s, x_1, y_1, t\rangle$. Then the flow is increased by r^0 and the residual capacities of four special edges become $0, r^1, r^2, r^2$.

[Flow augmenting step] Rename the special edges with residual capacities $0, r^n, r^{n+1}, r^{n+1}$ as A'_1, A'_2, A'_3, A'_4, respectively. (The edge set $\{A'_1, A'_2, A'_3, A'_4\}$ is just a rearrangement of the set of special edges $\{A_1, A_2, A_3, A_4\}$.) Repeat the following flow augmentations:

1. Choose a flow augmenting path p such that the special edges contained in p are only A'_2 and A'_3. For example, $p = \langle s, x'_2, y'_2, x'_3, y'_3, t\rangle$. ($x'_i$ and y'_i are the initial vertex and the terminal vertex of A'_i, respectively.)

2. Choose a flow augmenting path p such that p contains A'_2 as a forward edge and A'_1 and A'_3 as reverse edges. For example, $p = \langle s, x'_2, y'_2, y'_1, x'_1, y'_3, x'_3, y'_4, t\rangle$.

After the flow augmenting step, the residual capacities of the special edges A'_1, A'_2, A'_3, and A'_4 become $r^{n+2}, 0, r^{n+2}$, and r^{n+1}, respectively, and the value of the flow increases by $r^{n+1} + r^{n+2} = r^n$. (Note that *one* flow augmenting step in the procedure consists of *two* flow augmentations.)

Therefore, Procedure 1 does not terminate and the value of the flow converges to $S = \sum_{n=0}^{\infty} r^n = (\sqrt{5} + 3)/2$, which is not equal to the maximum flow $4S$.

Remark. All the previous examples [8], [13], [14], [15] also contains an edge with irrational capacity $r = (\sqrt{5} - 1)/2$ and simulate the computation of the recurrence $a_{n+2} = a_n - a_{n+1}$, except

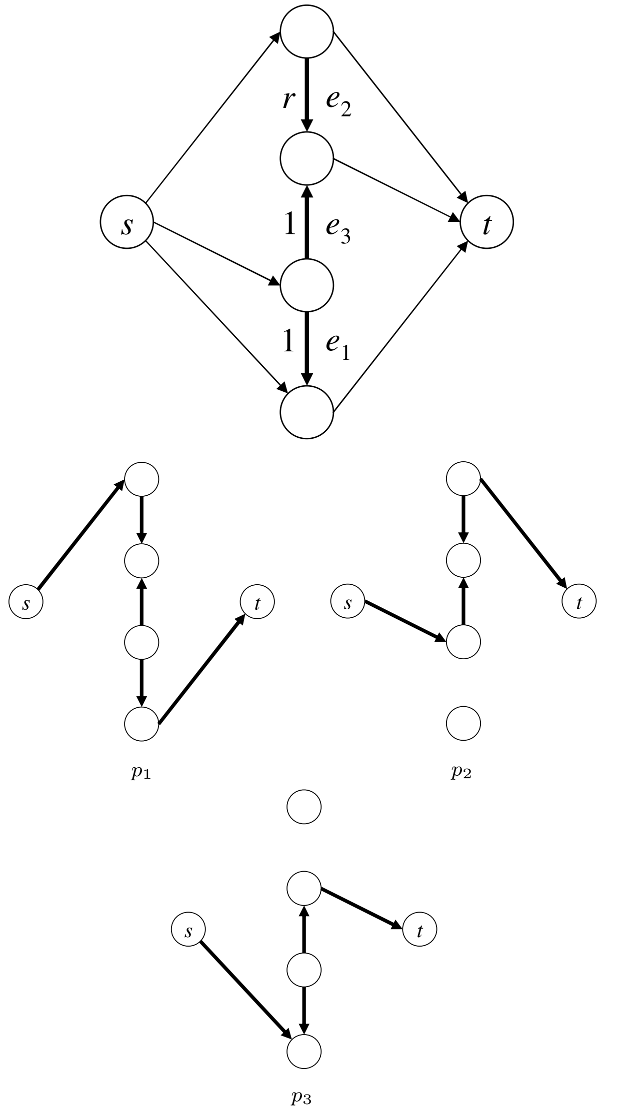


Fig. 2 Zwick's example N_1.

for Zwick's N_3.

3. Zwick's Examples

3.1 The Simplest Examples N_1 and N_2

N_1 simulates the computation of $a_{n+2} = a_n - a_{n+1}, a_0 = 1, a_1 = r$, where $r = (\sqrt{5} - 1)/2$. The special edges of N_1 are e_1, e_2, and e_3 whose capacities are $a_0 = 1, a_1 = r$, and $a_0 = 1$, respectively (Fig. 2). The other capacities are some integer $M \geq 4$. The following Procedure 2 does not terminate.

Procedure 2 (Zwick [13])

[Initial flow] Choose the flow augmenting path of length three from the source s to the sink t through e_3. The value of the flow is one and e_3 becomes saturated. The residual capacities of e_1, e_2, e_3 are $a_0, a_1, 0$, respectively. (In the following, we represent this situation in n-tuple $(a_0, a_1, 0)$.)

[Flow augmenting step] Let the current residual capacities of e_1, e_2, e_3 be $(a_n, a_{n+1}, 0)$. Repeat the following flow augmentations:

1. Choose the flow augmenting path p_1 (in Fig. 2). The residual capacities of special edges become $(a_{n+2}, 0, a_{n+1})$.

2. Choose the flow augmenting path p_2 (in Fig. 2). The residual

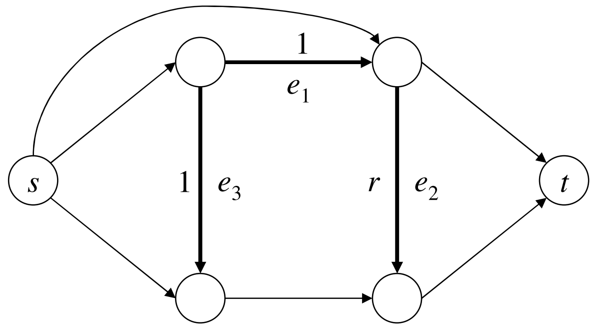


Fig. 3 Zwick's Example N_2.

capacities of special edges become $(a_{n+2}, a_{n+1}, 0)$.

3. Choose the flow augmenting path p_1 (in Fig. 2). The residual capacities of special edges become $(0, a_{n+3}, a_{n+2})$.

4. Choose the flow augmenting path p_3 (in Fig. 2). The residual capacities of special edges become $(a_{n+2}, a_{n+3}, 0)$.

The flow augmenting step increases the flow by $2a_{n+1}+2a_{n+2} = 2a_n$. Therefore, Procedure 2 does not terminate and the flow converges to $1 + 2\sum_{n=2}^{\infty} a_n = 3$, which is not equal to the maximum flow $2M + 1$. (Again, note that *one* flow augmenting step in the procedure consists of *four* flow augmentations.)

Network N_2 also has three special edges e_1, e_2, and e_3 which have capacities $a_0 = 1, a_1 = r$, and $a_1 = 1$, respectively (Fig. 3). The capacities of the other edges are $M \geq 4$. The non-termination of N_2 can be shown in a similar way.

3.2 The Smallest Example N_3

N_3 has only six vertices and eight edges. However, the capacities are determined by complicated processes: Four special edges e_1, e_2, e_3, and e_4 have capacities $1, r, r^2$, and 1, respectively, where $r = (1 + \sqrt{1-4\lambda})/2 \simeq 0.682378$ and $\lambda \simeq 0.216757$ is the only real root of $1 - 5x + 2x^2 - x^3 = 0$. The capacities of the other edges are some integer $M \geq 3$.

The irrational capacities r and r^2 are determined so that the following Procedure 3 does not terminate.

Procedure 3 (Zwick [13])

[Initial flow] Choose the flow augmenting path of length three from the source s to the sink t through e_4. The value of the flow is one and e_4 becomes saturated.

[Flow augmenting step] Let the current residual capacities of special edges e_1, e_2, e_3, and e_4 be $(x, y, z, 0)$, where $x > y > z > x - y > y - z$. Repeat the following flow augmentations:

1. Choose flow augmenting path p_1 (in Fig. 4). The residual capacities of the special edges become $(x - y, 0, z, y)$.

2. Choose flow augmenting path p_2 (in Fig. 4). The residual capacities of the special edges become$(x - y, z, 0, y - z)$.

3. Choose flow augmenting path p_3 (in Fig. 4). The residual capacities of the special edges become$(0, z - (x - y), x - y, y - z)$.

4. Choose flow augmenting path p_4 (in Fig. 4). The residual capacities of the special edges become$(y - z, z - (x - y), (x - y) - (y - z), 0)$.

After the flow augmenting step, the residual capacities x', y', and z' of e_1, e_2, and e_3 satisfy the following equation.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The characteristic polynomial of the above matrix is $1 - 5x + 2x^2 - x^3 = 0$ which has only one real root $\lambda \simeq 0.216757$. Here, $(1, r, r^2)$ is an eigenvector corresponding to the eigenvalue λ. Determine the initial flow so that the residual capacities of e_1, e_2, e_3, and e_4 become $(1, r, r^2, 0)$ at the beginning of the flow augmenting steps. Then after n iterations, the residual capacities become $\lambda^n \cdot (1, r, r^2, 0)$. The flow increases by $\lambda^{n-1}(1 + r)$ in the n-th iteration and converges to the maximum flow $1 + (1 + r)/(1 - \lambda) = 2 + r + r^2$.

4. Euclidean Algorithm and $N'_3(r)$

All the previous examples ever published have edges with special irrational capacities. In this section, we show that, for an arbitrary positive irrational number r, there exists a simplest and smallest example $N'_3(r)$ with only one edge irrational capacity of r.

The network topology of $N'_3(r)$ is the same as N_3. For an arbitrarily given irrational number $r > 0$, the capacities of $N'_3(r)$ are determined as follows: the capacities of two special edges e_1 and e_2 are 1 and r, respectively. The capacity of special edge e_4 is $c = \lceil 1 + r \rceil$. The capacities of the other five edges are $3c$. Note that e_3 is not a special edge.

Now, for $N'_3(r)$, the following theorem holds.

Theorem 1 $N'_3(r)$ is a simplest and smallest example of network on which the Ford-Fulkerson maximum flow procedure may fail to terminate. That is, $N'_3(r)$ is a simplest and smallest example of network which has an infinite sequence of flow augmentations. Moreover, the value of the flow converges to $c + 2(1 + r)$, which is not the maximum.

Proof. First, we show that there exists an infinite sequence of flow augmenting paths for $N'_3(r)$. The following procedure gives such a sequence.

Procedure 4

[Initial flow] Choose flow augmenting path $p_0 = \langle s, v_3, w_1, t \rangle$. The value of initial flow is c and the residual capacities of e_1, e_2, and e_4 become $(1, r, 0)$.

[Flow augmenting step] Let the current residual capacities of e_1, e_2, and e_4 be $(p, q, 0)$.

1. Choose augmenting path p_1 (in Fig. 4).

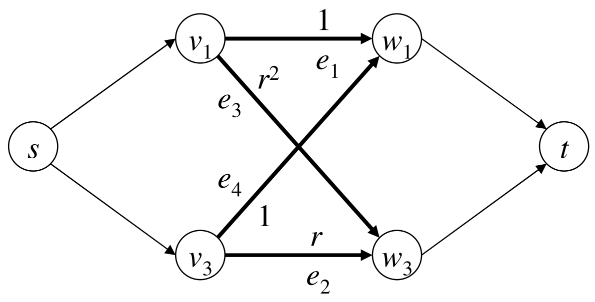
2. If $p > q$, the value of the flow increases by q and the residual capacities of e_1, e_2, e_4 are $(p - q, 0, q)$. Choose augmenting path p_2 (Fig. 4). The value of the flow increases by q again and the residual capacities of e_1, e_2, e_4 become $(p - q, q, 0)$.

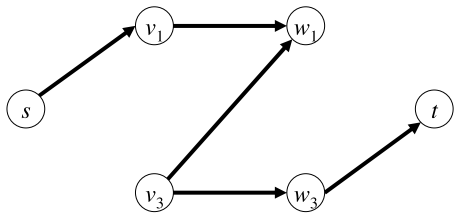
If $p < q$, the value of the flow increases by p and the residual capacities of e_1, e_2, e_4 are $(0, q - p, p)$. Choose augmenting path p_4 (Fig. 4). The value of the flow increases by p again and the residual capacities of e_1, e_2, e_4 become $(p, q - p, 0)$.

Now let

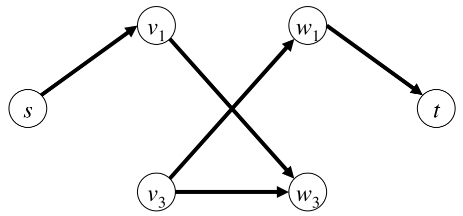
$$S_n = p_0(p_1p_2)^{a_0}(p_1p_4)^{a_1}(p_1p_2)^{a_2}(p_1p_4)^{a_3}\cdots(p_1p_k)^{a_n}$$

be the sequence of flow augmenting paths which have been generated at the end of $(a_0 + a_1 + \cdots a_n)$-th flow augmenting step in

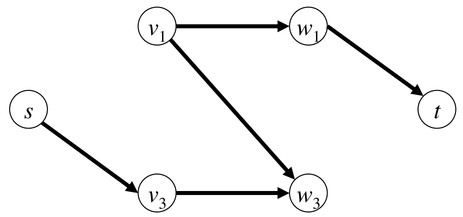




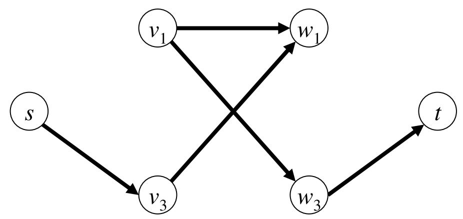
p_1



p_2



p_3



p_4

Fig. 4 Zwick's Example N_3.

Procedure 4, where $(p_1 p_i)^{a_j}$ denotes a_j time repetition of subsequence $p_1 p_i$ ($i = 2$ or 4) and $k = 2$ if n is even; otherwise $k = 4$.

It is easy to see that $a_0, a_1, \ldots, a_n$ are positive integers satisfying the following system of equations (*), except that $a_0 = 0$ if $r > 1$.

$$
(*)\quad
\begin{aligned}
x_0 &= 1, x_1 = r, \\
x_0 &= a_0 x_1 + x_2 && (0 < x_2 < x_1), \\
x_1 &= a_1 x_2 + x_3 && (0 < x_3 < x_2), \\
&\cdots \\
x_n &= a_n x_{n+1} + x_{n+2} && (0 < x_{n+2} < x_{n+1}).
\end{aligned}
$$

a_i and x_{i+2} are uniquely determined as the quotient and the remainder of division x_i by x_{i+1}. This procedure is none other than Euclidean algorithm.

The remainder x_i monotonically decreases, however, it cannot be zero; $x_i = 0$ implies that both x_0 and x_1 are integral multiples of x_{i-1}, which contradicts the assumption that r is irrational. Therefore, Procedure 4 never terminates and generates an infinite sequence of flow augmenting paths.

Let $f(S_n)$ be the value of the flow augmented by $S_n (n \geq 1)$. Then,

$$
\begin{aligned}
f(S_n) &= c + 2a_0 x_1 + 2a_1 x_2 + \cdots + 2a_n x_{n+1} \\
&= c + 2[(x_0 - x_2) + (x_1 - x_3) + \cdots + (x_n - x_{n+2})] \\
&= c + 2[(x_0 + x_1) - (x_{n+1} + x_{n+2})] \\
&= c + 2(1 + r) - 2(x_{n+1} + x_{n+2}).
\end{aligned}
$$

$\lim_{n\to\infty} f(S_n) = c + 2(1 + r)$ since $\lim_{n\to\infty} x_{n+1} + x_{n+2} = 0$. (For $i \geq 1$, $x_i = a_i x_{i+1} + x_{i+2} \geq x_{i+1} + x_{i+2} > 2x_{i+2}$. Then, $x_i/2 > x_{i+2}$. This shows that x_i decreases geometrically and $\lim_{n\to\infty} x_n = 0$.)

On the other hand, $N_3'(r)$ has two edge disjoint paths $P_a = \langle s, v_1, w_3, t\rangle$ and $P_b = \langle s, v_3, w_1, t\rangle$. We can send flows $3c$ and c along P_a and P_b, respectively. Therefore, the value of maximum flow is at least $4c$, which is larger than $\lim_{n\to\infty} f(S_n) = c + 2(1 + r) < 3c$. ■

Quotients a_i's in the Euclidean algorithm also appear in the continued fraction of $1/r$.

$$
\frac{1}{r} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}}}.
$$

Theorem 1 is also proved by the following theorem, which is a well-known result for continued fractions in elementary number theory. (See, for example, Chapter 2 of Ref. [3] or Chapter 5 of Ref. [16].)

Theorem 2 (1) An irrational number has a unique infinite continued fraction expansion.
(2) The value of an infinite continued fraction expansion is irrational.

[Example 1] Let the capacities of e_1 and e_2 be 1 and $r = (\sqrt{5} - 1)/2 < 1$, respectively. Then $1/r = (1 + \sqrt{5})/2$, which is the golden ratio. The infinite continued fraction of the golden ratio is:

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}.$$

Thus, the sequence of the flow augmenting paths given by Procedure 4 is p_0 followed by the infinite repetation of subsequence $p_1 p_2 p_1 p_4$ after p_0.

The following is known as Lagrange's theorem [3].

Theorem 3 x is a quadratic irrational if and only if its continued fraction is periodic. (x is quadratic irrational if it is a solution of a quadratic equation with integer coefficients.)

As a consequence of the theorem, the sequence of flow augmenting paths generated by Procedure 4 infinitely repeats some (finite) subsequence if and only if $1/r$ is quadratic irrational.

[Example 2] Let the capacities of e_1 and e_2 be 1 and $r = 1/\sqrt{3} < 1$, respectively. Then $1/r = \sqrt{3}$ and

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}.$$

Thus, the sequence of the flow augmenting paths given by Procedure 4 is $p_0 p_1 p_2$ followed by the infinite repetation of subsequence $p_1 p_4 p_1 p_2 p_1 p_2$.

5. Concluding Remarks

In this paper, we show a simplest and smallest network $N'_3(r)$ on which the Ford-Fulkerson maximum flow procedure may fail to terminate in the sense that it has an infinite sequence of flow augmentations. A major difference between $N'_3(r)$ and the previous examples is that the irrational edge capacity r can be arbitrarily given.

The result suggests that many networks with real-valued capacities might fail to terminate since networks with a certain number of vertices and edges may contains subgraphs homeomorphic to $N'_3(r)$ and the ratio of two random irrational edge capacities is irrational with probability one. (Let p and q are two random real numbers between 0 and 1. For fixed p, only countably many q's, which are the rational multiples of p, make the ratio p/q rational.)

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