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Balanced C_{20}-Bowtie Decomposition Algorithm of Complete Graphs

Kazuhiko Ushio Hideaki Fujimoto
Department of Informatics Depatrment of Electric and Electronic Engineering
Faculty of Science and Technology
Kinki University
ushio@info.kindai.ac.jp fujimoto@ele.kindai.ac.jp

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{20} be *the cycle* on 20 vertices. *The* C_{20}-*bowtie* is a graph of 2 edge-disjoint C_{20}'s with a common vertex and the common vertex is called *the center of the* C_{20}-*bowtie.* When K_n is decomposed into edge-disjoint sum of C_{20}-bowties, it is called that K_n *has a* C_{20}-*bowtie decomposition.* Moreover, when every vertex of K_n appears in the same number of C_{20}-bowties, it is called that K_n *has a balanced* C_{20}-*bowtie decomposition* and this number is called *the replication number.*

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_{20}-bowtie decomposition of K_n is $n \equiv 1 \pmod{80}$.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[8]. Horák and Rosa[2] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a* C_3-*bowtie system.*
In this sense, our balanced C_{20}-bowtie decomposition of K_n is to be known as *a balanced* C_{20}-*bowtie system.*

2. Balanced C_{20}-bowtie decomposition of K_n

Notation. We denote a C_{20}-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_{20} - v_1$, $v_1 - v_{21} - v_{22} - v_{23} - v_{24} - v_{25} - v_{26} - v_{27} - v_{28} - v_{29} - v_{30} - v_{31} - v_{32} - v_{33} - v_{34} - v_{35} - v_{36} - v_{37} - v_{38} - v_{39} - v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}), (v_1, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}, v_{37}, v_{38}, v_{39})\}$.

Theorem 1. K_n has a balanced C_{20}-bowtie decomposition if and only if $n \equiv 1 \pmod{80}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{20}-bowtie decomposition. Let b be the number of C_{20}-bowties and r be the replication number. Then $b = n(n-1)/80$ and $r = 39(n-1)/80$. Among r C_{20}-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{20}-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/80$ and $r_2 = 19(n-1)/40$. Therefore, $n \equiv 1 \pmod{80}$ is necessary.

(Sufficiency) $n = 80t + 1$. Construct tn C_{20}-bowties as follows:
$B_i^{(1)} = \{(i, i+1, i+4t+2, i+24t+2, i+36t+3, i+48t+3, i+68t+4, i+16t+3, i+60t+4, i+12t+3, i+54t+4, i+14t+3, i+64t+4, i+18t+3, i+72t+4, i+50t+3, i+40t+3, i+26t+2, i+8t+2, i+2t+1), (i, i+2, i+4t+4, i+24t+3, i+36t+5, i+48t+4, i+68t+6, i+16t+4, i+60t+6, i+12t+4, i+54t+6, i+14t+4, i+64t+6, i+18t+4, i+72t+6, i+50t+4, i+40t+5, i+26t+3, i+8t+4, i+2t+2)\}$
$B_i^{(2)} = \{(i, i+3, i+4t+6, i+24t+4, i+36t+7, i+48t+5, i+68t+8, i+16t+5, i+60t+8, i+12t+5, i+54t+8, i+14t+5, i+64t+8, i+18t+5, i+72t+8, i+50t+5, i+40t+7, i+26t+4, i+8t+6, i+2t+3), (i, i+4, i+4t+8, i+24t+5, i+36t+9, i+48t+6, i+68t+10, i+16t+6, i+60t+10, i+12t+6, i+54t+10, i+14t+6, i+64t+10, i+18t+6, i+72t+10, i+50t+6, i+40t+9, i+26t+5, i+8t+8, i+2t+4)\}$

...
$B_i^{(t)} = \{(i, i+2t-1, i+8t-2, i+26t, i+40t-1, i+50t+1, i+72t, i+18t+1, i+64t, i+14t+1, i+58t, i+16t+1, i+68t, i+20t+1, i+76t, i+52t+1, i+44t-1, i+28t, i+12t-2, i+4t-1), (i, i+2t, i+8t, i+26t+1, i+40t+1, i+50t+2, i+72t+2, i+18t+2, i+64t+2, i+14t+2, i+58t+2, i+16t+2, i+68t+2, i+20t+2, i+76t+2, i+52t+2, i+44t+1, i+28t+1, i+12t, i+4t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_{20}-bowtie decomposition of K_n.

Example 1. Balanced C_{20}-bowtie decomposition of K_{81}.

$B_i = \{(i, i+1, i+6, i+26, i+39, i+51, i+72, i+19, i+64, i+15, i+58, i+17, i+68, i+21, i+76, i+53, i+43, i+28, i+10, i+3), (i, i+2, i+8, i+27, i+41, i+52, i+74, i+20, i+66, i+16, i+60, i+18, i+70, i+22, i+78, i+54, i+45, i+29, i+12, i+4)\}$
$(i = 1, 2, ..., 81)$.

Example 2. Balanced C_{20}-bowtie decomposition of K_{161}.

$B_i^{(1)} = \{(i, i+1, i+10, i+50, i+75, i+99, i+140, i+35, i+124, i+27, i+112, i+31, i+132, i+39, i+148, i+103, i+83, i+54, i+18, i+5), (i, i+2, i+12, i+51, i+77, i+100, i+142, i+36, i+126, i+28, i+114, i+32, i+134, i+40, i+150, i+104, i+85, i+55, i+20, i+6)\}$
$B_i^{(2)} = \{(i, i+3, i+14, i+52, i+79, i+101, i+144, i+37, i+128, i+29, i+116, i+33, i+136, i+41, i+152, i+105, i+87, i+56, i+22, i+7), (i, i+4, i+16, i+53, i+81, i+102, i+146, i+38, i+130, i+30, i+118, i+34, i+138, i+42, i+154, i+106, i+89, i+57, i+24, i+8)\}$
$(i = 1, 2, ..., 161)$.

Example 3. Balanced C_{20}-bowtie decomposition of K_{241}.

$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+208, i+51, i+184, i+39, i+166, i+45, i+196, i+57, i+220, i+153, i+123, i+80, i+26, i+7), (i, i+2, i+16, i+75, i+113, i+148, i+210, i+52, i+186, i+40, i+168, i+46, i+198, i+58, i+222, i+154, i+125, i+81, i+28, i+8)\}$
$B_i^{(2)} = \{(i, i+3, i+18, i+76, i+115, i+149, i+212, i+53, i+188, i+41, i+170, i+47, i+200, i+59, i+224, i+155, i+127, i+82, i+30, i+9), (i, i+4, i+20, i+77, i+117, i+150, i+214, i+54, i+190, i+42, i+172, i+48, i+202, i+60, i+226, i+156, i+129, i+83, i+32, i+10)\}$
$B_i^{(3)} = \{(i, i+5, i+22, i+78, i+119, i+151, i+216, i+55, i+192, i+43, i+174, i+49, i+204, i+61, i+228, i+157, i+131, i+84, i+34, i+11), (i, i+6, i+24, i+79, i+121, i+152, i+218, i+56, i+194, i+44, i+176, i+50, i+206, i+62, i+230, i+158, i+133, i+85, i+36, i+12)\}$
$(i = 1, 2, ..., 241)$.

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