Balanced (C_4, C_9)-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4, C_9 be *the* 4-*cycle* and *the* 9-*cycle*, respectively. *The* (C_4, C_9)-$2t$-*foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_9's with a common vertex and the common vertex is called *the center of the* (C_4, C_9)-$2t$-*foil*. When K_n is decomposed into edge-disjoint sum of (C_4, C_9)-$2t$-foils, it is called that K_n *has a* (C_4, C_9)-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_9)-$2t$-foils, it is called that K_n *has a balanced* (C_4, C_9)-$2t$-*foil decomposition* and this number is called *the replication number*.

2. Balanced (C_4, C_9)-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_9)-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{26t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_9)-$2t$-foil decomposition. Let b be the number of (C_4, C_9)-$2t$-foils and r be the replication number. Then $b = n(n-1)/26t$ and $r = (11t+1)(n-1)/26t$. Among r (C_4, C_9)-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_9)-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/26t$ and $r_2 = 11(n-1)/26$. Therefore, $n \equiv 1 \pmod{26t}$ is necessary.

(Sufficiency) Put $n = 26st + 1$ and $T = st$.

Case 1. $T = 1$ **and** $n = 27$. **(Example 1.)**
Case 2. $T = 2$ **and** $n = 53$. **(Example 2.)**
Case 3. $T = 3$ **and** $n = 79$. **(Example 3.)**
Case 4. $T = 4$ **and** $n = 105$. **(Example 4.)**
Case 5. $T \geq 5$ **and** $n \geq 131$.

First, consider a sequence $S : g_1, g_2, g_3, ..., g_T$.
When $T \equiv 1 \pmod 4$, $T \geq 5$, put $T = 4p + 1$ and $S : g_1, g_2, g_3, ..., g_{4p+1}$ with
$S_1 : g_1, g_2, g_3, ..., g_{2p-1}$
$S_2 : g_{2p}$
$S_3 : g_{2p+1}, g_{2p+3}, g_{2p+5}, ..., g_{4p-1}$
$S_4 : g_{2p+2}, g_{2p+4}, g_{2p+6}, ..., g_{4p}$
$S_5 : g_{4p+1}$
such as
$S_1 : 14T + 4, 14T + 7, 14T + 10, ..., 14T + 6p - 2$
$S_2 : 14T + 6p + 2$
$S_3 : 14T + 6p + 6, 14T + 6p + 12, 14T + 6p + 18, ..., 17T - 3$
$S_4 : 14T + 6p + 5, 14T + 6p + 11, 14T + 6p + 17, ..., 17T - 4$
$S_5 : 17T$.
When $T \equiv 2 \pmod 4$, $T \geq 6$, put $T = 4p + 2$ and $S : g_1, g_2, g_3, ..., g_{4p+2}$ with
$S_1 : g_1, g_2, g_3, ..., g_{2p-2}$
$S_2 : g_{2p-1}, g_{2p}, g_{2p+1}, g_{2p+2}$
$S_3 : g_{2p+3}, g_{2p+5}, g_{2p+7}, ..., g_{4p+1}$
$S_4 : g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+2}$
such as
$S_1 : 14T + 4, 14T + 7, 14T + 10, ..., 14T + 6p - 5$
$S_2 : 14T + 6p - 1, 14T + 6p + 2, 14T + 6p + 3, 14T + 6p + 8$
$S_3 : 14T + 6p + 12, 14T + 6p + 18, 14T + 6p + 24, ..., 17T$
$S_4 : 14T + 6p + 11, 14T + 6p + 17, 14T + 6p + 23, ..., 17T - 1$.
When $T \equiv 3 \pmod 4$, $T \geq 7$, put $T = 4p + 3$ and $S : g_1, g_2, g_3, ..., g_{4p+3}$ with
$S_1 : g_1, g_2, g_3, ..., g_{2p}$
$S_2 : g_{2p+1}$
$S_3 : g_{2p+2}, g_{2p+4}, g_{2p+6}, ..., g_{4p+2}$
$S_4 : g_{2p+3}, g_{2p+5}, g_{2p+7}, ..., g_{4p+3}$
such as
$S_1 : 14T + 4, 14T + 7, 14T + 10, ..., 14T + 6p + 1$
$S_2 : 14T + 6p + 5$

S_3 : $14T+6p+9, 14T+6p+15, 14T+6p+21, ..., 17T$
S_4 : $14T+6p+8, 14T+6p+14, 14T+6p+20, ..., 17T-1$.
When $T \equiv 0 \pmod 4$, $T \geq 8$, put $T = 4p$ and $S : g_1, g_2, g_3, ..., g_{4p}$ with
$S_1 : g_1, g_2, g_3, ..., g_{2p-3}$
$S_2 : g_{2p-2}, g_{2p-1}, g_{2p}, g_{2p+1}$
$S_3 : g_{2p+2}, g_{2p+4}, g_{2p+6}, ..., g_{4p-2}$
$S_4 : g_{2p+3}, g_{2p+5}, g_{2p+7}, ..., g_{4p-1}$
$S_5 : g_{4p}$
such as
$S_1 : 14T+4, 14T+7, 14T+10, ..., 14T+6p-8$
$S_2 : 14T+6p-4, 14T+6p-1, 17T+3, 14T+6p+5$
S_3 : $14T+6p+9, 14T+6p+15, 14T+6p+21, ..., 17T-3$
S_4 : $14T+6p+8, 14T+6p+14, 14T+6p+20, ..., 17T-4$
$S_5 : 17T$.
Next, construct n (C_4, C_9)-$2T$-foils as follows:
$B_i = \{(i, i+18T+1, i+23T+2, i+6T+1), (i, i+1, i+3T+2, i+8T+2, i+18T+3, i+20T+3, i+3T+3, i+g_1, i+2T+1)\}$
$\cup \{(i, i+5T+2, i+23T+4, i+6T+2), (i, i+2, i+3T+4, i+8T+3, i+18T+5, i+20T+4, i+3T+5, i+g_2, i+2T+2)\}$
$\cup \{(i, i+5T+3, i+23T+6, i+6T+3), (i, i+3, i+3T+6, i+8T+4, i+18T+7, i+20T+5, i+3T+7, i+g_3, i+2T+3)\}$
$\cup$ ...
$\cup \{(i, i+6T, i+25T, i+7T), (i, i+T, i+5T, i+9T+1, i+20T+1, i+21T+2, i+5T+1, i+g_T, i+3T)\}$ $(i = 1, 2, ..., n)$.
Last, decompose each (C_4, C_9)-$2T$-foil into s (C_4, C_9)-$2t$-foils. Then they comprise a balanced (C_4, C_9)-$2t$-foil decomposition of K_n. ■

Example 1. A balanced (C_4, C_9)-2-foil decomposition of K_{27} .
$B_i = \{(i, i+6, i+25, i+7), (i, i+1, i+11, i+13, i+16, i+21, i+5, i+17, i+4)\}$
$(i = 1, 2, ..., 27)$.

Example 2. A balanced (C_4, C_9)-4-foil decomposition of K_{53}.
$B_i = \{(i, i+11, i+48, i+13), (i, i+1, i+20, i+23, i+28, i+38, i+6, i+31, i+8)\}$
$\cup \{(i, i+12, i+50, i+14), (i, i+2, i+22, i+26, i+32, i+41, i+10, i+34, i+7)\}$
$(i = 1, 2, ..., 53)$.

Example 3. A balanced (C_4, C_9)-6-foil decomposition of K_{79}.
$B_i = \{(i, i+55, i+71, i+19), (i, i+1, i+11, i+26, i+57, i+63, i+12, i+47, i+7)\}$
$\cup \{(i, i+17, i+73, i+20), (i, i+2, i+13, i+27, i+59, i+64, i+14, i+51, i+8)\}$
$\cup \{(i, i+18, i+75, i+21), (i, i+3, i+15, i+28, i+61, i+65, i+16, i+50, i+9)\}$
$(i = 1, 2, ..., 79)$.

Example 4. A balanced (C_4, C_9)-8-foil decomposition of K_{105}.
$B_i = \{(i, i+73, i+94, i+25), (i, i+1, i+38, i+46, i+55, i+72, i+8, i+68, i+16)\}$
$\cup \{(i, i+22, i+96, i+26), (i, i+2, i+40, i+47, i+57, i+75, i+12, i+71, i+15)\}$
$\cup \{(i, i+23, i+98, i+27), (i, i+3, i+42, i+48, i+59, i+78, i+21, i+64, i+14)\}$
$\cup \{(i, i+24, i+100, i+28), (i, i+4, i+44, i+49, i+61, i+81, i+20, i+67, i+13)\}$
$(i = 1, 2, ..., 105)$.

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