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$\hat{C}_k$ - factorization algorithm of symmetric complete multipartite digraphs

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1. Introduction

Let $K^*_{n_1,n_2,...,n_m}$ denote *the symmetric complete multipartite digraph* with partite sets $V_1, V_2, ..., V_m$ of $n_1, n_2, ..., n_m$ vertices each, and let $\hat{C}_k$ denote *the directed cycle* of length k on two partite sets. A spanning subgraph F of $K^*_{n_1,n_2,...,n_m}$ is called a $\hat{C}_k$ - *factor* if each component of F is $\hat{C}_k$. If $K^*_{n_1,n_2,...,n_m}$ is expressed as an arc-disjoint sum of $\hat{C}_k$ - factors, then this sum is called a $\hat{C}_k$ - *factorization* of $K^*_{n_1,n_2,...,n_m}$. In this paper, it is shown that a necessary and sufficient condition for the existence of such a factorization is (i) $k \equiv 0 \pmod 2$, $k \geq 4$, (ii) $n_1 = n_2 = ... = n_m \equiv 0 \pmod{k/2}$ for even m and $n_1 = n_2 = ... = n_m \equiv 0 \pmod k$ for odd m.

2. $\hat{C}_k$ - factorization of $K^*_{n_1,n_2,...,n_m}$

Notation. Given a $\hat{C}_k$ - factorization of $K^*_{n_1,n_2,...,n_m}$, let
r be the number of factors
b be the total number of components
t be the number of components of each factor
$t_{i,j}$ $(i < j)$ be the numbers of components whose vertices are in V_i and V_j among t components of each factor.
For a vertex x in V_i, let $r_{i,j}(x)$ be the numbers of components whose vertices are in V_i and V_j among r components having vertex x.

Theorem 1. If $K^*_{n_1,n_2,...,n_m}$ has a $\hat{C}_k$ - factorization, then (i) $k \equiv 0 \pmod 2$, $k \geq 4$, (ii) $n_1 = n_2 = ... = n_m \equiv 0 \pmod{k/2}$ for even m and $n_1 = n_2 = ... = n_m \equiv 0 \pmod k$ for odd m.
Proof. Suppose that $K^*_{n_1,n_2,...,n_m}$ has a $\hat{C}_k$ - factorization. Then $b = 2(n_1n_2 + n_1n_3 + ... + n_{m-1}n_m)/k, t = (n_1 + n_2 + ... + n_m)/k, r = b/t = 2(n_1n_2 + n_1n_3 + ... + n_{m-1}n_m)/(n_1 + n_2 + ... + n_m)$. $\hat{C}_k$ is a directed cycle of length k on two partite sets. Therefore, we have $k \equiv 0 \pmod 2$ and $k \geq 4$. For a vertex x in V_i, we have $r_{i,1} = n_1, r_{i,2} = n_2, ..., r_{i,i-1} = n_{i-1}, r_{i,i+1} = n_{i+1}, ..., r_{i,m} = n_m$ and $r_{i,1}+r_{i,2}+...+r_{i,i-1}+r_{i,i+1}+...+r_{i,m} = r$ $(i = 1, 2, ..., m)$. Put $n_1+n_2+...+n_m = N$. Then $N-n_1 = N - n_2 = ... = N - n_m = r$. Therefore, we have $n_1 = n_2 = ... = n_m$. Put $n_1 = n_2 = ... = n_m = n$. Then $b = m(m-1)n^2/k, t = mn/k, r = (m-1)n$. Put $t_{j,i} = t_{i,j} (i < j)$ and $t_{i,i} = 0$. Then, in a factor, $(t_{1,1} + t_{1,2} + ... + t_{1,m})k/2 = (t_{2,1} + t_{2,2} + ... + t_{2,m})k/2 = ... = (t_{m,1} + t_{m,2} + ... + t_{m,m})k/2 = n$. Put $t_i = t_{i,1} + t_{i,2} + ... + t_{i,m}$ $(i = 1, 2, ..., m)$. Then $t_1k/2 = t_2k/2 = ... = t_mk/2 = n$. Put $t_1 = t_2 = ... = t_m = T$. Then $T = 2n/k$.
Case m is even: Put $m = 2m', k = 2k'$. Then $b = m'(2m'-1)n^2/k', t = m'T, T = n/k', r = (2m'-1)n$. Therefore, we have $n \equiv 0 \pmod{k/2}$.
Case m is odd: Put $m = 2m'+1, k = 2k'$. Then $b = (2m'+1)m'n^2/k', t = m(T/2), T/2 = n/2k', T = n/k', r = 2m'n$. Therefore, we have $n \equiv 0 \pmod k$.

Notation. For a $\hat{C}_k : v_1 \to v_2 \to ... \to v_{k-1} \to v_k \to v_1$, we denote $\hat{C}_k(v_1, v_3, ..., v_{k-1}; v_2, v_4, ..., v_k)$.

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Theorem 2. When $k \equiv 0 \pmod 2, k \geq 4$ and $n \equiv 0 \pmod{k/2}$, $K^*_{n,n}$ has a $\hat{C}_k$ - factorization.

Proof. Put $n = ks/2$ and $h = k/2$. Let $V_1 = \{1, 2, ..., h\}$ and $V_2 = \{1', 2', ..., h'\}$. Construct h $\hat{C}_k$'s as following : $\hat{C}_k(1, 2, ..., h; 1', 2', ..., h')$, $\hat{C}_k(1, 2, ..., h; 2', 3', ..., h', 1')$, $\hat{C}_k(1, 2, ..., h; 3', ..., h', 1', 2')$,..., $\hat{C}_k(1, 2, ..., h; h', 1', 2', ..., (h-1)')$. Then they are $\hat{C}_k$ - factors of $K^*_{k/2,k/2}$, and they comprise a $\hat{C}_k$ - factorization of $K^*_{k/2,k/2}$. As a well-known result, $K_{s,s}$ has a 1-factorization. Therefore, $K^*_{ks/2,ks/2}$ has a $K^*_{k/2,k/2}$ - factorization. $K^*_{k/2,k/2}$ has a $\hat{C}_k$ - factorization as shown above. Thus $K^*_{n,n}$ has a $\hat{C}_k$ - factorization.

Theorem 3. When $k \equiv 0 \pmod 2$, $k \geq 4$, m is even and $n \equiv 0 \pmod{k/2}$, $K^*_{n,n,...,n}$ has a $\hat{C}_k$ - factorization.

Proof. Put $n = ks/2$. As a well-known result, K_m has a 1-factorization. So $K^*_{1,1,...,1}$ has a $K^*_{1,1}$ - factorization. Therefore, $K^*_{ks/2,ks/2,...,ks/2}$ has a $K^*_{ks/2,ks/2}$ - factorization. By Theorem 2, $K^*_{sk/2,ks/2}$ has a $\hat{C}_k$ - factorization. Thus $K^*_{n,n,...,n}$ has a $\hat{C}_k$ - factorization.

Theorem 4. When $k \equiv 0 \pmod 2$, $k \geq 4$, m is odd and $n \equiv 0 \pmod k$, $K^*_{n,n,...,n}$ has a $\hat{C}_k$ - factorization.

Proof. Put $n = ks$. As a well-known result, K_{2m} has a 1-factorization. $K_{2m} =$ 1-factor $\cup K_{2,2,...,2}$. So $K^*_{2,2,...,2}$ has a $K^*_{1,1}$ - factorization. Therefore, $K^*_{ks,ks,...,ks}$ has a $K^*_{ks/2,ks/2}$ - factorization. By Theorem 2, $K^*_{ks/2,ks/2}$ has a $\hat{C}_k$ - factorization. Thus $K^*_{n,n,...,n}$ has a $\hat{C}_k$ - factorization.

We have the following main theorem.

Main Theorem. $K^*_{n_1,n_2,...,n_m}$ has a $\hat{C}_k$ - factorization if and only if (i) $k \equiv 0 \pmod 2$, $k \geq 4$, (ii) $n_1 = n_2 = ... = n_m \equiv 0 \pmod{k/2}$ for even m and $n_1 = n_2 = ... = n_m \equiv 0 \pmod k$ for odd m.

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