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On the Church-Rosser Property of Root-E-overlapping and Strongly Depth-preserving Term Rewriting Systems

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A term rewriting system (TRS) is said to be strongly depth-preserving if for any rewrite rule and any variable appearing in its both sides, the minimal depth of the variable occurrences in the left-hand-side is greater than or equal to the maximal depth of the variable occurrences in the right-hand-side. This paper gives a sufficient condition for the Church-Rosser property of strongly depth-preserving TRS's and shows how to check this condition. By assigning a positive integer (called *weight*) to each function symbol, the notion of a strongly depth-preserving system is naturally extended to that of a strongly weight-preserving system, and a similar sufficient condition for the Church-Rosser property of strongly weight-preserving TRS's is obtained.

1. Introduction

A term rewriting system (TRS) is a set of directed equations (called rewrite rules). A TRS is Church-Rosser (CR) if any two interconvertible terms reduce to some common term by applications of the rewrite rules. This CR property is important in various applications of TRS's and has received much attention[1)~3),5)~8)]. Although the CR property is undecidable for general TRS's, many sufficient conditions for ensuring this property have been obtained[1),2),5)~8)].

However, for nonlinear and nonterminating TRS's, few results related to the CR property have been obtained. Our previous papers[5),6)] were the first to give nontrivial conditions for the CR property by using the notions of non-E-overlapping (stronger than nonoverlapping) and E-critical pairs to extend that of critical pairs. We gave some sufficient conditions for the CR property that can be applied to subclasses of right-linear TRS's. In the case of non-right-linear TRS's, it has been shown that there exist non-E-overlapping and depth-preserving TRS's that do not satisfy the CR property, but that all non-E-overlapping and strongly depth-preserving* TRS's satisfy the CR property[9)~11)]. Here, a TRS is depth-preserving if for each rule $\alpha \to \beta$ and any variable x appearing in both α and β, the maximal depth of the x occurrences in α is greater than or equal to that of the x occurrences in β[3)]. A TRS is strongly depth-preserving* if it is depth-preserving and for each rule $\alpha \to \beta$ and for any variable x appearing in α, all the depths of the x occurrences in α are the same[11)].

In this paper, we first slightly extend the definition of strongly depth-preserving* TRS's; that is, a TRS is strongly depth-preserving if for each rule $\alpha \to \beta$ and any variable x appearing in both α and β, the minimal depth of the x occurrences in α is greater than or equal to the maximal depth of the x occurrences in β. Obviously, a TRS is strongly depth-preserving if it is strongly depth-preserving*. We extend the result in Ref. 11) by showing that for this new class of strongly depth-preserving TRS's, the non-E-overlapping condition also ensures the CR property.

Next, we show that even if a strongly depth-preserving TRS is E-overlapping, a condition called root-E-closed (in Section 3) ensures the CR property (Theorem 1), and we give a decidable sufficient condition for this root-E-closed condition.

By assignment of a positive integer (called *weight*) to each function symbol, the notion of depth is naturally extended to that of weight: the weight of an x occurrence is the sum of the weights of function symbols appearing in the path from the root to the x-occurrence. Using the notion of this weight, we can give the definition of strongly weight-preserving TRS's in a similar way to that of strongly depth-preserving TRS's and obtain the corresponding root-E-closed condition which ensures the CR property of strongly weight-preserving TRS's (Theorem 2). For example, TRS $R = \{\, f(x) \to g(h(x), x),$ $g(x,x) \to a,\ b \to h(b)\}$[12)], where x is a variable and f, g, h, a, b are function symbols, is

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CR by this result (see Example 4 in Section 6). (It was stated in Middeldorp, et al.[12] without proof that this TRS R is CR.)

This paper is organized as follows. Section 2 is devoted to definitions. In Section 3, we give the root-E-closed condition. Some assertions to prove Theorem 1 are given in Section 4 and Theorem 1 is proven in Section 5. In Section 6, we give a sufficient condition for the CR property of strongly weight-preserving TRS's, and in Section 7, we give a decidable sufficient condition ensuring the root-E-closed condition.

2. Definitions

The following definitions and notations are similar to those in Refs. 2), 5) and 8) $\sim$ 11). We use ε to denote the empty string and ϕ to denote the empty set. Let X be a set of *variables*, F be a finite set of *function symbols* graded by an arity function $\mathsf{a} : F \to \{0, 1, 2, \cdots\}$, and T be the set of *terms* constructed from X and F.

For a term M, we define the set $O(M)$ of the *occurrences* (*positions*) as follows: If M is a variable, then $O(M) = \{\varepsilon\}$. If $M = f(M_1, M_2, \cdots, M_n)$ for some $f \in F$, $M_1, \cdots, M_n \in T$, then $O(M) = \{\varepsilon\} \cup \{iu \mid 1 \leq i \leq n,\ u \in O(M_i)\}$. For example, $O(f(g(x), a)) = \{\varepsilon, 1, 11, 2\}$ where $x \in X$, $f, g, a \in F$. We use M/u to denote the subterm of M at occurrence u, and $M[u \leftarrow N]$ to denote the term obtained from M by replacing the subterm M/u by term N. The set of occurrences $O(M)$ of M is partially ordered according to the prefix ordering: $u \leq v$ iff $\exists w.\ uw = v$. In this case, we denote w by v/u. If $u \leq v$ and $u \neq v$, then $u < v$. If $u \not\leq v$ and $v \not\leq u$, then u and v are said to be *disjoint*, and are denoted by $u|v$. Let $V(M)$ be the set of variables in M, $O_x(M)$ be the set of occurrences of variable $x \in V(M)$, and $O_X(M) = \cup_{x \in V(M)} O_x(M)$, that is, the set of variable occurrences in M. Let $\bar{O}(M) = O(M) - O_X(M)$, the set of non-variable occurrences. We also use $N[u \leftarrow M/u \mid u \in U]$ to denote the term $(N[u_1 \leftarrow M/u_1]) \cdots [u_n \leftarrow M/u_n]$ if $U = \{u_1, \cdots, u_n\}$ and $u_1, \cdots, u_n$ are pairwise disjoint. We use $root(M)$ to denote the function symbol of a term M at occurrence ε, that is, the top symbol.

The *depth* of occurrence $u \in O(M)$ is defined by $|u|$, i.e., the length of u. Let $H(M) = Max\{|u| \mid u \in O(M)\}$, the *height* of M. For example, $H(f(g(x))) = 2$, $H(a) = 0$, where $f, g, a \in F$ and $x \in X$.

A rewrite rule is a directed equation $\alpha \to \beta$ such that $\alpha \in T - X$, $\beta \in T$ and $V(\alpha) \supseteq V(\beta)$. A term rewriting system (TRS) R is a finite set of rewrite rules.

A term M reduces to a term N at occurrence u if $M/u = \sigma(\alpha)$ and $N = M[u \leftarrow \sigma(\beta)]$ for some $\alpha \to \beta \in R$ and $\sigma : X \to T$. We denote this reduction by $M \overset{u}{\to} N$. In this notation u may be omitted (i.e., $M \to N$) and $\to^*$ is the reflexive-transitive closure of $\to$. Let $M \overset{u}{\leftrightarrow} N$ be $M \overset{u}{\to} N$ or $N \overset{u}{\to} M$.

Let $\gamma : M_0 \overset{u_0}{\leftrightarrow} M_1 \cdots M_{n-1} \overset{u_{n-1}}{\leftrightarrow} M_n$ be a sequence of $\leftrightarrow$ reductions. Then, let $R(\gamma) = \{u_0, \cdots, u_{n-1}\}$. If $R(\gamma)$ are pairwise disjoint, then γ is said to be a two-way parallel reduction, and is denoted by $M_0 \leftrightarrow\!\!\!\!\!\leftrightarrow M_n$. Let $R(M_0 \leftrightarrow\!\!\!\!\!\leftrightarrow M_n) = R(\gamma)$. (Note: $R(\gamma) = \phi$, i.e., $M_0 \leftrightarrow\!\!\!\!\!\leftrightarrow M_0$ is allowed.) The term 'two-way' can be omitted if there is no possibility of confusion. Let $\leftrightarrow\!\!\!\!\!\leftrightarrow^*$ be the reflexive-transitive closure of $\leftrightarrow\!\!\!\!\!\leftrightarrow$. If $u < v$ for all $v \in R(M \leftrightarrow\!\!\!\!\!\leftrightarrow N)$, then we denote this parallel reduction by $M \overset{>u}{\leftrightarrow\!\!\!\!\!\leftrightarrow} N$. In particular, if $u = \varepsilon$, then we may use $M \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\!\leftrightarrow} N$. Let $\overset{>u}{\leftrightarrow\!\!\!\!\!\leftrightarrow}{}^*$ and $\overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\!\leftrightarrow}{}^*$ be the reflexive-transitive closures of $\overset{>u}{\leftrightarrow\!\!\!\!\!\leftrightarrow}$ and $\overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\!\leftrightarrow}$, respectively.

We assume that $\gamma : M_0 \leftrightarrow\!\!\!\!\!\leftrightarrow M_1 \leftrightarrow\!\!\!\!\!\leftrightarrow \cdots \leftrightarrow\!\!\!\!\!\leftrightarrow M_n$ in the following definitions.

Let $R(\gamma) = \bigcup_{0 \leq i < n} R(M_i \leftrightarrow\!\!\!\!\!\leftrightarrow M_{i+1})$ and $MR(\gamma)$ be the set of *minimal* occurrences in $R(\gamma)$ under the prefix ordering. For $u \in O(M_0)$, if there exists no $v \in R(\gamma)$ such that $v \leq u$, then γ is said to be *u-invariant*. Let $M_0 = \sigma(\alpha)$ or $M_n = \sigma(\alpha)$ for some $\alpha \to \beta \in R$ and $\sigma : X \to T$. Then, γ is said to be *α-keeping* if γ is u-invariant for all $u \in \bar{O}(\alpha)$. That is, γ is α-keeping iff all reductions of γ occur in the variable parts of α. Parallel reduction sequence γ is said to be a *peak* if $\gamma : M_0 \overset{\varepsilon}{\leftarrow} M_1 \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\!\leftrightarrow}{}^* M_{n-1} \overset{\varepsilon}{\to} M_n$. We denote by $\gamma[i, j]$ the subsequence $M_i \leftrightarrow\!\!\!\!\!\leftrightarrow M_{i+1} \leftrightarrow\!\!\!\!\!\leftrightarrow \cdots \leftrightarrow\!\!\!\!\!\leftrightarrow M_j$ of γ where $0 \leq i \leq j \leq n$. Let $u \in MR(\gamma)$. Then, the *cut* sequence of γ at u is $\gamma/u = (M_0/u \leftrightarrow\!\!\!\!\!\leftrightarrow M_1/u \leftrightarrow\!\!\!\!\!\leftrightarrow \cdots \leftrightarrow\!\!\!\!\!\leftrightarrow M_n/u)$. We denote by $\gamma[\xi'/\xi]$ the sequence obtained from reduction sequence γ by replacing subsequence or cut sequence (or cut subsequence) ξ of γ by sequence ξ'. The composition of γ and $\delta : N_0 \leftrightarrow\!\!\!\!\!\leftrightarrow N_1 \leftrightarrow\!\!\!\!\!\leftrightarrow \cdots \leftrightarrow\!\!\!\!\!\leftrightarrow N_k$, where $N_0 = M_n$, is denoted by $(\gamma; \delta)$.

Let γ^R be the reverse sequence of γ, namely, $\gamma^R : M_n \leftrightarrow\!\!\!\!\!\leftrightarrow \cdots \leftrightarrow\!\!\!\!\!\leftrightarrow M_1 \leftrightarrow\!\!\!\!\!\leftrightarrow M_0$. The number of parallel reduction steps of γ is $|\gamma|_p = n$. (Note: If $\delta : M \leftrightarrow\!\!\!\!\leftrightarrow M$, then $|\delta|_p = 1$.) Let $net(\gamma)$ be the sequence obtained from γ by removing all $M_i \leftrightarrow\!\!\!\!\leftrightarrow M_{i+1}$ that satisfy the equation $M_i = M_{i+1}$, $0 \le i < n$. We use $|\gamma|_{np}$ to denote $|net(\gamma)|_p$. Let $H(\gamma) = Max\{H(M_i) \mid 0 \le i \le n\}$.

Example. Let $\delta : f(c,c) \leftrightarrow\!\!\!\!\leftrightarrow f(g(c), g(c)) \leftrightarrow\!\!\!\!\leftrightarrow a \leftrightarrow\!\!\!\!\leftrightarrow a$, then $|\delta|_p = 3$, $net(\delta) : f(c,c) \leftrightarrow\!\!\!\!\leftrightarrow f(g(c), g(c)) \leftrightarrow\!\!\!\!\leftrightarrow a$, $|\delta|_{np} = 2$ and $H(\delta) = H(f(g(c), g(c))) = 2$.

We need the definitions of $left(\gamma, h)$, $right(\gamma, h)$, $ldis(\gamma, h)$ and $width(\gamma, h)$ in Ref. 11): Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$. Then, $left(\gamma, h)$ is i $(0 \le i \le n)$ if $H(M_i)$ is h and $H(M_0), \cdots, H(M_{i-1})$ are less than h, and $left(\gamma, h)$ is undefined if there exists no such M_i; $right(\gamma, h)$ is defined in a similar way to $left(\gamma, h)$ by replacing the term "left" with "right"; that is, $|\gamma|_p - left(\gamma^R, h)$; $ldis(\gamma, h)$ is $|\gamma|_p - left(\gamma, h)$; $width(\gamma, h)$ is defined if either $left(\gamma, h)$ or $right(\gamma, h)$ is defined, and in this case, $width(\gamma, h)$ is $right(\gamma, h') - left(\gamma, h'')$ where h' (resp. h'') is the least value satisfying the conditions that $h' \ge h$ (resp. $h'' \ge h$) and $right(\gamma, h')$ (resp. $left(\gamma, h'')$) is defined. Roughly speaking, $left(\gamma, h)$ returns the suffix i of M_i (i.e., the ith term of γ) that is first found as the term of height h in searching from the left end of γ, while $right(\gamma, h)$ returns that found in searching from the right end, and $width(\gamma, h)$ is $right(\gamma, h) - left(\gamma, h)$. We also need the definitions of $K_{ldis}(\gamma)$, $K_{right}(\gamma)$ and $K_{width}(\gamma)$ in Ref. 11): Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$ and $\vec{H}(\gamma) = (H(M_0), H(M_1), \cdots, H(M_n))$. Intuitively, we can suppose that the sequence of the heights $\vec{H}(\gamma)$ represents the shape of a mountain; that is to say, $H(M_i)$ is the height of the mountain slope at a distance i from the left end, $0 \le i \le n$. To characterize $\vec{H}(\gamma)$, we use the following three parameters $K_{ldis}(\gamma)$, $K_{right}(\gamma)$ and $K_{width}(\gamma)$: $K_{ldis}(\gamma)$ is the set of $|\gamma|_p - left(\gamma, h)$, $0 \le h \le H(\gamma)$, and gives a characterization of the shape of the left slope of the mountain represented by $\vec{H}(\gamma)$; $K_{right}(\gamma)$ is the set of $right(\gamma, h)$, $0 \le h \le H(\gamma)$, and gives a characterization of the shape of the right slope of the mountain; $K_{width}(\gamma)$ is the set of $right(\gamma, h) - left(\gamma, h)$, $0 \le h \le H(\gamma)$, and gives the widths of the mountain at the lines of height h, $0 \le h \le H(\gamma)$. Henceforth, $\vec{H}(\gamma)$ is called a mountain. These formal definitions are given in Ref. 11).

Example. Let $\delta : f(c) \leftrightarrow\!\!\!\!\leftrightarrow f(g(g(c))) \leftrightarrow\!\!\!\!\leftrightarrow f(g(c)) \leftrightarrow\!\!\!\!\leftrightarrow f(f(g(g(c)))) \leftrightarrow\!\!\!\!\leftrightarrow f(f(c)) \leftrightarrow\!\!\!\!\leftrightarrow g(c)$ where $f, g, c \in F$. Then, we have $left(\delta, 1) = 0$, $left(\delta, 3) = 1$, $left(\delta, 4) = 3$, $right(\delta, 1) = 5$, $right(\delta, 2) = 4$, $right(\delta, 4) = 3$, $ldis(\delta, 1) = 5$, $ldis(\delta, 3) = 4$, $ldis(\delta, 4) = 2$, $width(\delta, 1) = 5$, $width(\delta, 2) = 3$, $width(\delta, 3) = 2$, $width(\delta, 4) = 0$. We have $K_{ldis}(\delta) = \{(1,5), (3,4), (4,2)\}$, $K_{width}(\delta) = \{(1,5), (2,3), (3,2), (4,0)\}$ and $K_{right}(\delta) = \{(1,5), (2,4), (4,3)\}$.

We define an ordering $<_s \subseteq N \times N$ (where $N = \{0, 1, 2, \cdots\}$) as follows: $(a, b) <_s (a', b') \Leftrightarrow (a < a' \bigwedge b \le b') \bigvee (a = a' \bigwedge b < b')$. Let $\le_s$ be $<_s \bigcup =$. We use $\ll_s$ to denote the multiset ordering of this ordering $<_s$. Let $\lleqq_s$ be $\ll_s \bigcup =$. We use $\{\cdots\}_m$ to denote a multiset, e.g., $\{1, 1, 2\}_m$. We use $\ll_w$ to denote the multiset ordering of a lexicographic ordering $<$ (i.e., $(a, b) < (a', b') \Leftrightarrow (a < a') \bigvee (a = a' \bigwedge b < b')$). Let $\lleqq_w$ be $\ll_w \bigcup =$. Note that if $(a, b) <_s (a', b')$, then $(a, b) < (a', b')$, but the converse does not necessarily hold. If $A \ll_s B$, then $A \ll_w B$. The orderings of $>_s$ and $>$ are well-founded, and thus $\gg_s$ and $\gg_w$ are well-founded[1]. These relations $\ll_w$ and $\ll_s$ are used as well-founded orderings for both sets $K_{ldis}(\gamma)$'s, $K_{right}(\gamma)$'s, $K_{width}(\gamma)$'s and for multisets of them such as $K_{ldis}(\gamma) \cup K_{ldis}(\gamma')$.

$K_{ldis}(\gamma)$, $K_{right}(\gamma)$ and $K_{width}(\gamma)$ are used to prove assertions $S(n)$, $P(k)$ and $Q(k)$. Moreover, $K_{ldis}(\gamma)$ is used in the definition of the root-E-closed condition. $\ll_s$ will be used to prove assertion $S(n)$ and and $\ll_w$ will be used to prove assertions $P(k)$ and $Q(k)$.

Definition of $\langle \delta \preceq \gamma \rangle$

We define a relation $\preceq$ over parallel reduction sequences as follows. Let $\gamma : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ and $\delta : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$. Then, $\delta \preceq \gamma$ if $|\delta|_p = |\gamma|_p$, $|\delta|_{np} \le |\gamma|_{np}$, $K_{ldis}(\delta) \lleqq_s K_{ldis}(\gamma)$ and $K_{right}(\delta) \lleqq_s K_{right}(\gamma)$. (Note that if $\delta \preceq \gamma$, then $H(\delta) \le H(\gamma)$ holds, since $K_{ldis}(\delta) \lleqq_s K_{ldis}(\gamma)$, and $\preceq$ is reflexive and transitive.) Intuitively speaking, $\delta \preceq \gamma$ if mountain $\vec{H}(\delta)$ is screened by mountain $\vec{H}(\gamma)$, when $\vec{H}(\gamma)$ is put in front of $\vec{H}(\delta)$, i.e., $\vec{H}(\delta)$ is not a bigger mountain than $\vec{H}(\gamma)$. This ordering $\preceq$ will be used to prove $S(n)$ and define the root-E-closed condition.

A pair of rewrite rules $\alpha \to \beta$ and $\alpha' \to \beta'$ is *overlapping* iff there exist $u \in \bar{O}(\alpha)$ and mappings $\sigma, \sigma' : X \to T$ such that $\sigma(\alpha/u) = \sigma'(\alpha')$, where $u = \varepsilon$ implies that $(\alpha \to \beta) \neq (\alpha' \to \beta')$. In this case, the pair is overlapping at u, and *root-overlapping* if $u = \varepsilon$.

Definitions of ⟨E-overlapping TRS, root-E-overlapping TRS⟩[8)]

Two rules $\alpha \to \beta$ and $\alpha' \to \beta'$ are said to be E-overlapping at $u \in \bar{O}(\alpha)$ iff there exists a sequence $\sigma(\alpha/u) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} \sigma'(\alpha')$ for some mappings σ and σ'. A TRS R is said to be E-overlapping if there exist two rewrite rules that are E-overlapping in R. Note that in this definition, if $\sigma(\alpha/u) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} \sigma'(\alpha')$ is replaced by $\sigma(\alpha/u) = \sigma'(\alpha')$, then the two rules are said to be overlapping[2)], and R is overlapping. If R is overlapping, then R is also E-overlapping, but the converse does not necessarily hold. A TRS is said to be root-E-overlapping if every two rewrite rules overlap at ε (the root position) when they are E-overlapping.

An E-overlapping parallel reduction sequence $\gamma : \sigma(\alpha)[u \leftarrow \sigma'(\beta')] \leftarrow \sigma(\alpha)[u \leftarrow \sigma'(\alpha')] \overset{>u}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} \sigma(\alpha) \to \sigma(\beta)$ is *standard* iff for the subsequence $\gamma' : \sigma'(\alpha') \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* \sigma(\alpha/u)$, $R(\gamma') \cap (\bar{O}(\alpha') \cap \bar{O}(\alpha/u)) = \phi$; that is, any reduction in γ' occurs in the variable parts in $\sigma'(\alpha')$ or $\sigma(\alpha/u)$.

Definition of ⟨strongly depth-preserving TRS⟩

TRS R is *strongly depth-preserving* if $\forall \alpha \to \beta \in R \ \ \forall x \in V(\beta) \quad Max\{|v| \mid v \in O_x(\beta)\} \leq Min\{|u| \mid u \in O_x(\alpha)\}$.

That is, a TRS is said to be strongly depth-preserving if for any rewrite rule and any variable appearing in its both sides, the minimal depth of the variable occurrences in the left-hand-side is greater than or equal to the maximal depth of the variable occurrences in the right-hand-side.

Note. This definition slightly extends the previous one[9)~11)] (i.e., TRS R was said to be strongly depth-preserving if $\forall \alpha \to \beta \in R \ \forall x \in V(\beta) \ Max\{|v| \mid v \in O_x(\beta)\} \leq Max\{|u| \mid u \in O_x(\alpha)\}$ and $\forall x \in V(\alpha) \, \forall u, v \in O_x(\alpha) \ |u| = |v|$.)

Example. Let $R_1 = \{f(x,x) \to a,\ c \to g(c),\ g(x) \to f(x,x)\}$ and $R_2 = \{f(g(x), g(g(x))) \to h(g(x), g(c)),\ c \to g(c)\}$ where $a, c, f, g \in F$ and $x \in X$. Both R_1 and R_2 are strongly depth-preserving. (But R_2 was not strongly depth-preserving in the previous definition[11)].)

3. Root-E-closed Condition

In this section, we introduce a condition called root-E-closed. We will prove that this is a sufficient condition for the CR property of root-E-overlapping and strongly depth-preserving TRS's.

Definition of ⟨root-E-closed TRS⟩

A TRS R is *root-E-closed* if R is root-E-overlapping and satisfies the following condition (*):

(*) For any standard root-E-overlapping parallel reduction sequence $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} \sigma'(\alpha') \to \sigma'(\beta')$ for some rule $\alpha \to \beta$, $\alpha' \to \beta' \in R$ and mappings $\sigma, \sigma' : X \to T$, there exists $\delta : \sigma(\beta) \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* \sigma'(\beta')$ such that the following conditions (i) and (ii) hold:

(i) $\delta \preceq \gamma$

(ii) At least one of the following conditions (1)–(3) holds:

(1) $|\delta|_{np} < |\gamma|_{np}$

(2) $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$

(3) If $\delta[0,1] : \sigma(\beta) \overset{\varepsilon}{\leftarrow} M$ for some M, then $\delta[1, |\delta|_p] : M \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* \sigma'(\beta')$ is ε-invariant.

Note that if $\delta \preceq \gamma$ and $H(\delta) < H(\gamma)$, then (2) holds.

That is, a TRS is said to be root-E-closed if it is root-E-overlapping and for every $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} \sigma'(\alpha') \to \sigma'(\beta')$, there exists a parallel reduction sequence $\delta : \sigma(\beta) \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* \sigma'(\beta')$ such that $\delta \preceq \gamma$; that is, $\vec{H}(\delta)$ is screened by $\vec{H}(\gamma)$ and δ strictly less than γ under one of the components of $\preceq$, i.e., $|\delta|_{np} < |\gamma|_{np}$ or $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$.

Example 1. TRS $R_3 = \{f(x,x) \to h(x,x),\ f(g(x),x) \to a,\ c \to g(c),\ h(g(x),x) \to a\}$, where $a, c, f, g, h \in F$ and $x \in X$, is strongly depth-preserving.

We first note that only the pair of the first and second rules has E-overlapping sequences. Thus, R_3 is root-E-overlapping. Let $\gamma : h(\sigma(x), \sigma(x)) \overset{\varepsilon}{\leftarrow} f(\sigma(x), \sigma(x)) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} f(g(\sigma'(x)), \sigma'(x)) \overset{\varepsilon}{\to} a$ be a standard root-E-overlapping sequence. Using $\sigma(x) \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* g(\sigma'(x))$ and $\sigma(x) \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^* \sigma'(x)$, let $\delta : h(\sigma(x), \sigma(x)) \overset{\varepsilon-inv}{\longleftrightarrow\!\!\!\!\!\!\longleftrightarrow^*} h(g(\sigma'(x)), \sigma'(x)) \to a \longleftrightarrow\!\!\!\!\!\!\longleftrightarrow a$. Then, $|\delta|_p = |\gamma|_p$ and $|\delta|_{np} = |\gamma|_{np} - 1$, and thus part (1) of the root-E-closed condition (ii) holds, that is, $|\delta|_{np} < |\gamma|_{np}$. It is straightforward to show that $\delta \preceq \gamma$. Hence, R_3 is root-E-closed.

4. Assertions

In this section, we show that if TRS R is root-E-closed and strongly depth-preserving, then R is CR. Henceforth, we are dealing with a fixed TRS R and assume that R is root-E-closed and strongly depth-preserving unless otherwise stated.

To show this, we use the following five assertions: $S(n)$, $P(k)$, $P'(k)$, $Q(k)$ and $Q'(k)$ (where $n \geq 0$, $k \geq 0$). Assertion $Q(k)$ ensures that TRS R is CR.

Assertion $S(n)$

Let $\gamma : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ where $|\gamma|_p = n$.

Then, $\exists\delta : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ such that the following conditions (i) and (ii) hold:

(i) There are no peaks in δ

(ii) $\delta \preceq \gamma$

Assertion $P(k)$

Let $\gamma : M \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\leftrightarrow^*} \sigma(\alpha) \to \sigma(\beta)$ for some rule $\alpha \to \beta \in R$ and mapping σ where $H(\gamma) \leq k$.

Then, $\exists\delta : N \leftrightarrow\!\!\!\!\leftrightarrow^* \sigma(\beta)$ for some N such that the following conditions (i) and (ii) hold:

(i) $M \to^* N$

(ii) Either $H(\delta) < H(\gamma)$ or δ is ε-invariant and $H(\delta) = H(\gamma)$.

Assertion $P'(k)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \leftrightarrow\!\!\!\!\leftrightarrow M_2 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$ where $H(\gamma) \leq k$, the number of ε-reductions in γ is l (> 0) and each ε-reduction is $M_i \xrightarrow{\varepsilon} M_{i+1}$ for some i $(0 \leq i < n)$. Let $M_{i_1} \xrightarrow{\varepsilon} M_{i_1+1}, \cdots, M_{i_l} \xrightarrow{\varepsilon} M_{i_l+1}$ be the ε-reductions of γ, $0 \leq i_1 < i_2 \cdots < i_l < n$. Then, there exist i_j $(1 \leq j \leq l)$ and $\delta : N \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_j+1}$ for some N such that the following conditions (i) and (ii) hold:

(i) $M_0 \to^* N$

(ii) Either $H(\delta) < H(\gamma[0, i_j + 1])$ holds or $i_j = i_l$, $H(\delta) = H(\gamma[0, i_j + 1])$ and δ is ε-invariant.

Assertion $Q(k)$[11)]

Let $\gamma : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ where $H(\gamma) \leq k$. Then, $\exists\delta : M \leftrightarrow\!\!\!\!\leftrightarrow^* L \leftrightarrow\!\!\!\!\leftrightarrow^* N$ for some L such that $H(\delta) \leq k$, $M \to^* L$ and $N \to^* L$.

Assertion $Q'(k)$[11)]

Let $\gamma_i : M \leftrightarrow\!\!\!\!\leftrightarrow^* M_i$, where $H(\gamma_i) \leq k$, $1 \leq i \leq n$ and $n \geq 2$. Then, $\exists\, \delta : M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ for some N such that $H(\delta) \leq k$ and $\forall i$ $(1 \leq i \leq n)$ $M_i \to^* N$.

$S(n)$ says that for a given sequence γ, there exists δ such that δ has no peaks and $\delta \preceq \gamma$. On the other hand, the previous definitions of $S(n)$ and $S'(n)$ in Ref. 11) said that it is possible to remove the outermost peak, so that by repeating this process, we can obtain sequence δ having no peaks. The current version of $S(n)$ describes and extends this result. $P(k)$ and $P'(k)$ are slightly simpler than the versions given in Ref. 11) in the sense that the conditions that $M \leftrightarrow\!\!\!\!\leftrightarrow^* N$ and $M_0 \leftrightarrow\!\!\!\!\leftrightarrow^* N$ had to satisfy in the previous definitions are removed. $Q(k)$ and $Q'(k)$ are the same as in Ref. 11).

To prove these assertions, we use the following properties of $ldis$, $right$ and $width$ given in Ref. 11).

Property 3[11)]

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$. Let $u \in MR(\gamma)$ and $\bar{\gamma} = \gamma[i,j]/u$ where $0 \leq i < j \leq n$. Let $\delta : L_i \leftrightarrow\!\!\!\!\leftrightarrow L_{i+1} \cdots \leftrightarrow\!\!\!\!\leftrightarrow L_j$ where $L_i = M_i/u, L_j = M_j/u$, $|\delta|_p = |\bar{\gamma}|_p$ and $H(\delta) \leq H(\bar{\gamma})$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$.

(1) If $K_{ldis}(\delta) \leqq_s K_{ldis}(\bar{\gamma})$, then $K_{ldis}(\gamma') \leqq_s K_{ldis}(\gamma)$.

(2) If $K_{right}(\delta) \leqq_s K_{right}(\bar{\gamma})$, then $K_{right}(\gamma') \leqq_s K_{right}(\gamma)$.

(3) If $K_{ldis}(\delta) \leqq_s K_{ldis}(\bar{\gamma})$ and $K_{right}(\delta) \leqq_s K_{right}(\bar{\gamma})$, then $K_{width}(\gamma') \leqq_s K_{width}(\gamma)$.

Property 4[11)]

Let γ be a parallel reduction sequence. Then, $K_{ldis}\ (net(\gamma)) \leqq_s K_{ldis}(\gamma)$ and $K_{right}(net(\gamma)) \leqq_s K_{right}(\gamma)$.

Property 5[11)]

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$ and $\bar{\gamma} = \gamma[0, i]$ where $0 \leq i \leq n$. Let $\delta : L_0 \leftrightarrow\!\!\!\!\leftrightarrow L_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow L_j$ where $0 \leq j$, $L_j = M_i$ and $H(\delta) < H(\bar{\gamma})$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$. Then, $K_{ldis}(\gamma') \ll_w K_{ldis}(\gamma)$ and $K_{width}(\gamma') \ll_w K_{width}(\gamma)$.

Property 6

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$. Let $u \in MR(\gamma)$ and $\bar{\gamma} = \gamma[i,j]/u$ where $0 \leq i < j \leq n$. Let $\delta : L_i \leftrightarrow\!\!\!\!\leftrightarrow L_{i+1} \cdots \leftrightarrow\!\!\!\!\leftrightarrow L_j$ where $L_i = M_i/u, L_j = M_j/u$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$.

If $\delta \preceq \bar{\gamma}$, then $\gamma' \preceq \gamma$.

Property 7

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_l$ and $\delta : N_0 \leftrightarrow\!\!\!\!\leftrightarrow N_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow N_l$ where $\forall i$ $(0 \leq i \leq l)$ $\exists j, j'$ $(0 \leq j \leq i \leq j' \leq l)$ $H(N_i) \leq H(M_j)$ and $H(N_i) \leq H(M_{j'})$.

Then, $K_{ldis}(\delta) \leqq_s K_{ldis}(\gamma)$ and $K_{right}(\delta) \leqq_s K_{right}(\gamma)$. Moreove, if $|\delta|_{np} \leq |\gamma|_{np}$, then $\delta \preceq \gamma$.

We only need to prove Properties 6 and 7. See Ref. 11) for the proofs of the other properties.

Proof of Property 6

Note that $|\delta|_p = |\bar{\gamma}|_p$ holds because $\delta \preceq \bar{\gamma}$. If $K_{ldis}(\delta) \leqq_s K_{ldis}(\bar{\gamma})$ and $|\delta|_p = |\bar{\gamma}|_p$, then $K_{ldis}(\gamma') \leqq_s K_{ldis}(\gamma)$ holds by Property 3 (1). Similarly, $K_{right}(\gamma') \leqq_s K_{right}(\gamma)$ holds by Property 3 (2). It is obvious that $|\gamma'|_p = |\gamma|_p$ and $|\gamma'|_{np} \leq |\gamma|_{np}$. Thus, $\gamma' \preceq \gamma$ holds, and therefore Property 6 holds. □

Proof of Property 7

By Property 2 in Ref. 11), for any $Y \in \{ldis, right, width\}$ $K_Y(\delta) \leqq_s K_Y(\gamma)$ iff $\forall(h,l) \in K_Y(\delta)\ \exists(h',l') \in K_Y(\gamma)\ (h,l) \leq_s (h',l')$. Thus, Property 7 obviously holds. □

5. Proofs of Assertions

We are now ready to prove these assertions. We first prove $S(n)$ by induction on $n \geq 0$. Then we will prove that $P(k) \Rightarrow P'(k)$ and $Q(k) \Rightarrow Q'(k)$. Using these results, we will finally prove $P(k) \bigwedge Q(k)$ by induction on $k \geq 0$.

Proof of $S(n)$

We prove $S(n)$ by induction on $|\gamma|_p = n$. Let $\gamma : M \leftrightarrow\!\!\!\rightarrow^* N$ such that $|\gamma|_p = n$.

Basis. $n = 0, 1$.Obvious.

Induction Step. $n > 1$.

Let $weight_1(\gamma) = (H(\gamma), |\gamma|_{np}, K_{ldis}(\gamma))$ where we use $\ll_s$ as the ordering of $K_{ldis}(\gamma)$'s and use the lexicographic ordering $<$ as the ordering of $weight_1(\gamma)$'s. Note that if $\delta \preceq \gamma$, then obviously $weight_1(\delta) \leq weight_1(\gamma)$.

Basis. $H(\gamma) = 0$ and $|\gamma|_{np} = 0$. Obvious.

Induction Step.

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_n$. We apply the induction hypothesis to $\gamma[1,n] : M_1 \leftrightarrow\!\!\!\rightarrow^* M_n$. Then, there exists $\delta_1 : M_1 \leftrightarrow\!\!\!\rightarrow^* M_n$ such that there are no peaks in δ_1 and $\delta_1 \preceq \gamma[1,n]$. Let

$\delta = (M_0 \leftrightarrow\!\!\!\rightarrow M_1); \delta_1$.

Note that $\delta \preceq \gamma$ holds by $\delta_1 \preceq \gamma[1,n]$ and Property 6.

There exist two cases for $\gamma[0,1]$, namely, $M_0 \xrightarrow{\varepsilon} M_1$ or $M_0 \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\rightarrow} M_1$ (Case 1), and $M_0 \xleftarrow{\varepsilon} M_1$ (Case 2).

Case 1: $M_0 \xrightarrow{\varepsilon} M_1$ or $M_0 \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\rightarrow} M_1$

If $M_0 \xrightarrow{\varepsilon} M_1$, then there are no peaks in δ, since there are no peaks in δ_1. Thus, $S(n)$ holds.

Next, we consider the case of $M_0 \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\rightarrow} M_1$. Either δ_1 is ε-invariant (Case 1-1) or δ_1 has ε-reductions (Case 1-2).

Case 1-1: δ_1 is ε-invariant.

In this case, since δ is ε-invariant, we can apply the induction hypothesis to cut sequences δ/i of δ, where $1 \leq i \leq \mathsf{a}(root(M_0))$. That is, there exists η_i such that there are no peaks in η_i and $\eta_i \preceq \delta/i$. Let $\eta = \delta[\eta_1/(\delta/1)] \cdots [\eta_l/(\delta/l)]$ where $l = \mathsf{a}(root(M_0))$. Then, there are no peaks in η and $\eta \preceq \delta$ $(\preceq \gamma)$ holds by Property 6. Hence, $S(n)$ holds.

Case 1-2: δ_1 has ε-reductions.

Let $\delta_1 = \delta_{11}; \delta_{12}; \delta_{13}$ where δ_{11} is ε-invariant and δ_{12} is ε-reduction. We apply the induction hypothesis to $\delta'_{11} = ((M_0 \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\rightarrow} M_1); \delta_{11})$ and obtain ξ_1 such that there are no peaks in ξ_1, $\xi_1 \preceq \delta'_{11}$ and ξ_1 is ε-invariant. It follows that $\xi = (\xi_1; \delta_{12}; \delta_{13})$ has no peaks (since R is root-E-overlapping) and $\xi \preceq \delta(\preceq \gamma)$. The proof is similar to that in Case 1-1. Hence, $S(n)$ holds.

Case 2: $M_0 \xleftarrow{\varepsilon} M_1$.

There are two cases for δ_1: δ_1 is ε-invariant (Case 2-1) or δ_1 has ε-reductions (Case 2-2).

Case 2-1: δ_1 is ε-invariant.

In this case, δ has no peaks, and therefore $S(n)$ holds.

Case 2-2: δ_1 has ε-reductions.

We prove this case by induction on the number of ε-reductions of form $\xrightarrow{\varepsilon}$ in δ_1.

Basis. δ_1 has no ε-reductions of form $\xrightarrow{\varepsilon}$.

In this case, every ε-reduction in δ_1 is $\xleftarrow{\varepsilon}$, and thus δ has no peaks. Hence, $S(n)$ holds.

Induction Step. δ_1 has ε-reductions of form $\xrightarrow{\varepsilon}$.

Let $\delta_1 = \delta_{11}; \delta_{12}; \delta_{13}$ where δ_{11} is an ε-invariant and δ_{12} is ε-reduction. Then, $\delta_{12} : N \xrightarrow{\varepsilon} N'$ for some N, N', since if $N \xleftarrow{\varepsilon} N'$, then δ_1 could have a peak, which is a contradiction.

Let $\xi_1 = (M_0 \xleftarrow{\varepsilon} M_1); \delta_{11}; \delta_{12}$ where $M_0 = \sigma(\beta), M_1 = \sigma(\alpha)$ and $\delta_{12} : \sigma'(\alpha') \xrightarrow{\varepsilon} \sigma'(\beta')$ for some rules $\alpha \to \beta$, $\alpha' \to \beta'$ and mappings σ, σ'.

There are two cases: in Case 2-2-1, $(\alpha \to \beta) = (\alpha' \to \beta')$, and in Case 2-2-2, $(\alpha \to \beta) \neq (\alpha' \to \beta')$.

Case 2-2-1: $(\alpha \to \beta) = (\alpha' \to \beta')$.

Since δ_{11} satisfies the condition (i) of $S(n)$, that is, there are no peaks in δ_{11}, and TRS R is root-E-overlapping, we have $MR(\delta_{11}) \cap \bar{O}(\alpha) = \phi$.

Let $|\delta_{11}|_p = m$. For each $x \in V(\beta)$, we choose a redex occurrence $u_x \in O_x(\alpha)$. Let σ_k be mappings satisfying the equation $\sigma_k(x) = M_k/u_x$, $1 \leq k \leq m+1$. Let $N_k = \sigma_k(\beta)$ where $1 \leq k \leq m+1$. Then, we have $\xi'_1 : N_1 (= \sigma(\beta)) \leftrightarrow\!\!\!\rightarrow N_1 \leftrightarrow\!\!\!\rightarrow N_2 \leftrightarrow\!\!\!\rightarrow N_3 \cdots \leftrightarrow\!\!\!\rightarrow N_m \leftrightarrow\!\!\!\rightarrow N_{m+1} \leftrightarrow\!\!\!\rightarrow N_{m+1}$ $(= \sigma'(\beta))$. Note that $|\xi'_1|_p = m + 2 = |\xi_1|_p$ and $|\xi'_1|_{np} \leq |\xi_1|_{np} - 2 < |\xi_1|_{np}$ hold. It is straightforward to prove that $K_{ldis}(\xi'_1) \leqq_s K_{ldis}(\xi_1)$ and $K_{right}(\xi'_1) \leqq_s K_{right}(\delta')$ hold, since $H(N_k) \leq max(H(M_k), H(\beta))$, $1 \leq k \leq m+1$ hold by the strongly depth-preserving property. Hence, ξ'_1 satisfies the conditions that

$\xi'_1 \preceq \xi_1$ and $|\xi'_1|_{np} < |\xi_1|_{np}$.

We have $weight_1(\xi'_1; \delta_{13}) < weight_1(\delta)$ and $(\xi'_1; \delta_{13}) \preceq \delta$ by Property 6. (Note that $\delta = (\xi_1; \delta_{13})$.) Thus, by the induction hypothesis, $S(n)$ holds.

Case 2-2-2: $(\alpha \to \beta) \neq (\alpha' \to \beta')$.

Note that ξ_1 is standard, since δ_1 has no peaks and TRS R is root-E-overlapping. By the

root-E-closed condition, there exists $\xi_1' : \sigma(\beta) \leftrightarrow\!\!\!\!\leftrightarrow^* \sigma'(\beta')$ such that
(i) $\xi_1' \preceq \xi_1$ and
(ii) at least one of the following conditions (1)–(3) holds:
(1) $|\xi_1'|_{np} < |\xi_1|_{np}$
(2) $K_{ldis}(\xi_1') \ll_s K_{ldis}(\xi_1)$
(3) If $\xi_1'[0,1] : \sigma(\beta) \overset{\varepsilon}{\leftarrow} M$ for some M, then $\xi_1'[1, |\xi_1'|_p] : M \leftrightarrow\!\!\!\!\leftrightarrow^* \sigma'(\beta')$ is ε-invariant.

Note that $(\xi_1'; \delta_{13}) \preceq \delta$ by Property 6. If (1) or (2) holds, then $weight_1(\xi_1'; \delta_{13}) < weight_1(\delta)$. Thus, by the induction hypothesis, $S(n)$ holds.

There remains the case in which (3) holds. In this case, if $\xi_1'[0,1] : \sigma(\beta) \overset{\varepsilon}{\leftarrow} M$ for some M, then the number of $\overset{\varepsilon}{\rightarrow}$ reductions in $(\xi_1'; \delta_{13})$ is less than that in δ. Thus, the induction hypothesis ensures that $S(n)$ holds for $(\xi_1'; \delta_{13})$. Hence, $S(n)$ holds for γ.

If $\xi_1'[0,1]$ is $\sigma(\beta) \overset{\varepsilon}{\rightarrow} M$ or $\sigma(\beta) \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\leftrightarrow} M$ for some M, then we can easily prove that $S(n)$ holds: if $\sigma(\beta) \overset{\varepsilon}{\rightarrow} M$, then $(\xi_1'; \delta_{13})$ has no peaks, and thus $S(n)$ holds. If $\sigma(\beta) \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\leftrightarrow} M$, then we apply the induction hypothesis to $(\xi_1'[1, |\xi_1'|_p]; \delta_{13})$ and use the proof of Case 1, and consequently $S(n)$ holds. □

Proof of $P(k) \Rightarrow P'(k)$

We prove $P'(k)$ by induction on the number $l \geq 1$ of ε-reductions appearing in γ where $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_1} \overset{\varepsilon}{\rightarrow} M_{i_1+1} \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_2} \overset{\varepsilon}{\rightarrow} M_{i_2+1} \cdots \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_l} \overset{\varepsilon}{\rightarrow} M_{i_l+1} \leftrightarrow\!\!\!\!\leftrightarrow^* M_n$ and $H(\gamma) \leq k$.

Basis: the case of $l = 1$. Obvious.

Induction Step: the case of $l > 1$.

We first apply $P(k)$ to $\gamma[0, i_1 + 1]$, so that there exists $\delta : N \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_1+1}$ for some N such that

(i) $M_0 \rightarrow^* N$ (*p'1)
(ii) Either $H(\delta) < H(\gamma[0, i_1 + 1])$ or $H(\delta) = H(\gamma[0, i_1 + 1])$ and δ is ε-invariant. (*p'2)

If $H(\delta) < H(\gamma[0, i_1+1])$ holds, then let $j = 1$, that is, $i_j = i_1$, so that the above conditions (i) and (ii) ensure $P'(k)$.

The case in which $H(\delta) = H(\gamma[0, i_1 + 1])$ and δ is ε-invariant remains. Let $\gamma' = (\delta; \gamma[i_1 + 1, n]) : N \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_1+1} \leftrightarrow\!\!\!\!\leftrightarrow^* M_n$. Note that $H(\gamma') \leq k$ by (*p'2). Since δ is ε-invariant, γ' contains $(l - 1)$ ε-reductions: $M_{i_2} \overset{\varepsilon}{\rightarrow} M_{i_2+1}, \cdots, M_{i_l} \overset{\varepsilon}{\rightarrow} M_{i_l+1}$, and thus the induction hypothesis ensures that i_j $(2 \leq j \leq l)$ and $\eta : N' \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_j+1}$ exist for some N' such that the following conditions hold:

$N \rightarrow^* N'$ (*p'3)
Either $H(\eta) < H(\gamma'')$ where $\gamma'' = (\delta; \gamma[i_1 + 1, i_j + 1]) : N \leftrightarrow\!\!\!\!\leftrightarrow^* M_{i_j+1}$, or $H(\eta) = H(\gamma'')$, $i_j = i_l$ and η is ε-invariant. (*p'4)

Then, we can show that η satisfies the following required conditions:

$M_0 \rightarrow^* N'$ (*p'5)
Either $H(\eta) < H(\gamma[0, i_j + 1])$ or $H(\eta) = H(\gamma[0, i_j + 1])$, $i_j = i_l$ and η is ε-invariant. (*p'6)

By (*p'1) and (*p'3), we have $M_0 \rightarrow^* N'$, and thus (*p'5) holds. (*p'6) holds by (*p'4) and $H(\gamma'') \leq H(\gamma[0, i_j + 1])$. Hence, $P'(k)$ holds. □

Proof of $Q(k) \Rightarrow Q'(k)$

The proof is the same as that of $Q(k) \Rightarrow Q'(k)$ in Ref. 11), since the latter proof can be applied to any TRS. □

Proof of $P(k) \bigwedge Q(k)$

We prove that $P(k) \bigwedge Q(k)$ by induction on $k \geq 0$. We first prove $P(k)$, then $Q(k)$.

Basis: the case of $k = 0$.

Proof of $P(0)$

In this case, for $\gamma : M \leftrightarrow\!\!\!\!\leftrightarrow^* \sigma(\alpha) \rightarrow \sigma(\beta)$, note that $H(\gamma) = 0$ implies that β is not a variable, since $\alpha \in F$ (i.e., α is a function symbol) and $\sigma(\alpha) = M$ since $\bar{\gamma} : M \leftrightarrow\!\!\!\!\leftrightarrow^* \sigma(\alpha)$ is ε-invariant. Let $N = \sigma(\beta)$ and $\delta : N$. Then, $H(\delta) = 0, M \rightarrow^* N$ and δ is ε-invariant, as claimed. □

Proof of $Q(0)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$ where $n \geq 0$, $H(\gamma) = 0$, $M_0 = M$ and $M_n = N$. We prove $Q(0)$ by induction on $n \geq 0$. In the case in which $n = 0$ or $n = 1$, the proof is obvious. Consider the case of $n > 1$. Without loss of generality, we can assume that $M_i \neq M_{i+1}$ for all i $(0 \leq i < n)$. Thus, note that $M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1$ implies that $M_0 \overset{\varepsilon}{\rightarrow} M_1$ or $M_1 \overset{\varepsilon}{\rightarrow} M_0$ holds, since $H(\gamma) = 0$. If $M_0 \overset{\varepsilon}{\rightarrow} M_1$ holds, then by applying the induction hypothesis to $\gamma[1, n] : M_1 \leftrightarrow\!\!\!\!\leftrightarrow M_2 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_n$, $Q(0)$ holds for γ. The case of $M_1 \overset{\varepsilon}{\rightarrow} M_0$ remains. Similarly, for $M_i \leftrightarrow\!\!\!\!\leftrightarrow M_{i+1}$ where $1 \leq i < n$, $M_i \overset{\varepsilon}{\rightarrow} M_{i+1}$ or $M_{i+1} \overset{\varepsilon}{\rightarrow} M_i$ holds. If $M_{i+1} \overset{\varepsilon}{\rightarrow} M_i$ for all i $(1 \leq i < n)$, then $\gamma : M_0 \leftarrow M_1 \cdots \leftarrow M_n$, so that obviously $Q(0)$ holds. Otherwise, let i be the least number such that $M_i \overset{\varepsilon}{\rightarrow} M_{i+1}$ $(1 \leq i < n)$. Then $M_{i-1} \overset{\varepsilon}{\leftarrow} M_i \overset{\varepsilon}{\rightarrow} M_{i+1}$. If $M_{i-1} = M_{i+1}$, then the proof is obvious.

Otherwise, since TRS R is root-E-closed, $\exists\delta$: $M_{i-1} \leftarrow\!\!\!+\!\!\!\rightarrow^* M_{i+1}$ such that

(i) $\delta \preceq \gamma[i-1,i+1]$

(ii) At least one of the following conditions (1)–(3) holds:

(1) $|\delta|_{np} < |\gamma[i-1,i+1]|_{np}$

(2) $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma[i-1,i+1])$

(3) If $\delta[0,1] : M_{i-1} \overset{\varepsilon}{\leftarrow} M$ for some M, then $\delta[1,|\delta|_p] : M \leftarrow\!\!\!+\!\!\!\rightarrow^* M_{i+1}$ is ε-invariant.

By (i), $H(\delta) = 0$ and $|\delta|_p = |\gamma[i-1,i+1]|_p$ hold, and therefore neither (2) nor the case of $M_{i-1} \overset{\varepsilon}{\leftarrow} M$ in (3) holds. If (1) holds, then when the induction hypothesis is applied to $\gamma[0,i-1];\delta;\gamma[i+1,n]$, $Q(0)$ holds. The case of $M_{i-1} \overset{\varepsilon}{\rightarrow} M$ remains. If $i=1$, then $M_0 \overset{\varepsilon}{\rightarrow} M$ and $M \leftarrow\!\!\!+\!\!\!\rightarrow M_2 \leftarrow\!\!\!+\!\!\!\rightarrow^* M_n$. In this case, we are done. If $i > 1$, then we have $M_{i-2} \overset{\varepsilon}{\leftarrow} M_{i-1} \overset{\varepsilon}{\rightarrow} M$. If this process is repeated, (1) will hold in the end, or else $M_0 \overset{\varepsilon}{\rightarrow} M'$ for some M' such that $M' \leftarrow\!\!\!+\!\!\!\rightarrow M_2$, and thus $Q(0)$ will hold. □

Induction Step: the case of $k > 0$.

Proof of $P(k)$

Let $\gamma : M_0 \leftarrow\!\!\!+\!\!\!\rightarrow M_1 \cdots \leftarrow\!\!\!+\!\!\!\rightarrow M_{n-1} \leftarrow\!\!\!+\!\!\!\rightarrow M_n$ where $H(\gamma) = k$, $M_0 = M$, $M_{n-1} = \sigma(\alpha)$ and $M_n = \sigma(\beta)$. Let $\bar{\gamma} = \gamma[0,n-1]$ and $\Gamma_\gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i$ for some $u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$.

We define the weight of $\bar{\gamma}$ as follows:

$$weight_2(\bar{\gamma}) = \bigsqcup_{\gamma_i \in \Gamma_\gamma} K_{ldis}(net(\gamma_i))$$

where $\bigsqcup$ denotes the union of multisets. We use the multiset ordering $\ll_w$ as the ordering of $weight_2(\gamma)$'s and $\ll_w$ as the ordering of $K_{ldis}(\gamma)$'s.

Basis: the case of $weight_2(\bar{\gamma}) = \phi$, that is, $\Gamma_\gamma = \phi$ (in this case, $\bar{\gamma}$ is α-keeping).

For any $x \in V(\alpha)$, let $O_x(\alpha) = \{u_{x_1}, u_{x_2}, \cdots, u_{x_{l_x}}\}$. Since the reductions of γ occur only in the variable parts $\sigma(x)$'s, $x \in V(\alpha)$, we have

$\zeta_i : \sigma(x)\ (= M_{n-1}/u_{x_i}) \leftarrow\!\!\!+\!\!\!\rightarrow M_{n-2}/u_{x_i} \cdots \leftarrow\!\!\!+\!\!\!\rightarrow M_0/u_{x_i}\ (= M/u_{x_i})$

for all $u_{x_i} \in O_x(\alpha)$, $1 \le i \le l_x$ and $x \in V(\alpha)$. Note that $\zeta_i = (\bar{\gamma}/u_{x_i})^R$ and $H(\zeta_i) \le H(\gamma) - |u_{x_i}| = k - |u_{x_i}| < k$.

Let $h_x = Max\{H(\zeta_i) \mid 1 \le i \le l_x\}$. Let $u_x = u_{x_i}$ satisfy the condition that $|u_{x_i}| = Min\{|u_{x_1}|, \cdots, |u_{x_{l_x}}|\}$. Then,

$$h_x + |u_x| \le k \qquad (*\text{p1})$$

holds, since if $H(\zeta_j) = h_x$ for some j, then $h_x + |u_{x_j}| \le k$ and $|u_x| \le |u_{x_j}|$ hold. Hence, since $h_x < k$, we can apply $Q'(h_x)$ to $\zeta_1, \cdots, \zeta_{l_x}$ by the induction hypothesis. Thus, there exists $\eta_x : \sigma(x) \leftarrow\!\!\!+\!\!\!\rightarrow^* N_x$ for some term N_x such that

$$\begin{array}{ccccc} & & \gamma & & \\ M & \leftarrow\!\!\!+\!\!\!\rightarrow^* & \sigma(\alpha) & \rightarrow & \sigma(\beta) \\ & \searrow^* & & \swarrow\!\!\!\!\!\nearrow^* & \delta \\ & & \sigma'(\beta)(=N) & & \end{array}$$

Fig. 1 Parallel reduction sequences in the proof of $P(k)$.

$H(\eta_x) \le h_x$ and

$$\forall i\ (1 \le i \le l_x)\ \ M/u_{x_i} \rightarrow^* N_x. \qquad (*\text{p2})$$

Let $\sigma'(x) = N_x$ for all $x \in V(\alpha)$. Using $\eta_x^R : \sigma'(x) \leftarrow\!\!\!+\!\!\!\rightarrow^* \sigma(x)$, let

$$\delta : \sigma'(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* \sigma(\beta).$$

Then, the strongly depth-preserving property ensures that for all $x \in V(\beta)$, $v_x \in O_x(\beta)$, $|v_x| \le |u_x|$, so that we have

$$H(\delta) \le Max(\{h_x + |v_x| \mid v_x \in O_x(\beta) \cup \{H(\beta)\}\}) \le Max(h_x + |u_x|, H(\beta))$$

By (*p1), it follows that

$$H(\delta) \le k. \qquad (*\text{p3})$$

In particular,

if β is a variable, then $H(\delta) < k$. (*p4)

(To obtain $H(\delta) \le k$, we need the strongly depth-preserving condition, which is used only to establish this.) Since $M = \alpha[u_{x_i} \leftarrow M/u_{x_i} \mid u_{x_i} \in O_x(\alpha),\ x \in V(\alpha)]$ and by (*p2), there exists a parallel reduction sequence such that

$$M \rightarrow^* \sigma'(\alpha) \rightarrow \sigma'(\beta). \qquad (*\text{p5})$$

(See **Fig. 1**.)

Let $N = \sigma'(\beta)$. We can now prove that the conditions (i) and (ii) of $P(k)$ hold for $\delta : N \leftarrow\!\!\!+\!\!\!\rightarrow^* \sigma(\beta)$. Since $M \rightarrow^* N$ holds by (*p5), the condition (i) holds.

If β is a variable, then the condition (ii) holds by (*p4). Otherwise, δ is ε-invariant, and therefore condition (ii) holds by (*p3). Hence, $P(k)$ holds.

Induction Step: the case of $weight_2(\bar{\gamma}) \gg_w \phi$, that is, $\bar{\gamma}$ is not α-keeping. (Here, $\gamma : M_0(= M) \leftarrow\!\!\!+\!\!\!\rightarrow M_1 \cdots \leftarrow\!\!\!+\!\!\!\rightarrow M_{n-1}(= \sigma(\alpha)) \rightarrow M_n(= \sigma(\beta))$ and $\bar{\gamma} = \gamma[0,n-1]$.)

Let $\gamma_1 = \bar{\gamma}/u_1$ where $u_1 \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)$. Note that there exists an ε-reduction in γ_1, and that $H(\gamma_1) < k$. Let $\gamma'_1 = net(\gamma_1)$.

If $S(n)$ is applied to γ'_1, there exists a parallel reduction sequence $\eta : M_0/u_1 \leftarrow\!\!\!+\!\!\!\rightarrow^* M_{n-1}/u_1$ such that the following conditions (i) and (ii) hold:

(i) There are no peaks in η.

(ii) $\eta \preceq \gamma'_1$

Let $\gamma' = \bar{\gamma}[\eta/\gamma_1]$. $K_{ldis}(\eta) \underline{\ll}_s K_{ldis}(\gamma'_1)$ holds by (ii), and thus by Property 4,

$$weight_2(\gamma') \underline{\ll}_w weight_2(\bar{\gamma}) \qquad (*\text{p6})$$

(Note that $A \ll_s B$ implies that $A \ll_w B$.) Let $\eta' = net(\eta) : L_0 \leftrightarrow\!\!\!\rightarrow L_1 \cdots \leftrightarrow\!\!\!\rightarrow L_m$ where $m \le n-1$, $L_0 = M_0/u_1$, $L_m = M_{n-1}/u_1$.

Then, $L_i \overset{\varepsilon}{\leftarrow} L_{i+1}$ does not hold for any i $(0 \le i < m)$ by (i) and root-E-overlapping of R.

If η' is ε-invariant, then the definition of $weight_2$ ensures that $weight_2(\gamma') \ll_w weight_2(\bar{\gamma})$, and thus $P(k)$ holds for $(\gamma'; (M_{n-1} \leftrightarrow\!\!\!\rightarrow M_n))$ by the induction hypothesis. Hence, $P(k)$ also holds for γ. Let η' have l (> 0) ε-reductions, that is, let $L_{i_1} \overset{\varepsilon}{\rightarrow} L_{i_1+1}, \cdots, L_{i_l} \overset{\varepsilon}{\rightarrow} L_{i_l+1}$ be the ε-reductions where $0 \le i_1 < \cdots < i_l < m$. Then, by the induction hypotheses $P(k')$ and $P'(k')$ for $k' < k$ (since $P(k') \Rightarrow P'(k')$), there exists i_j $(1 \le j \le l)$ and $\zeta : N \leftrightarrow\!\!\!\rightarrow^* L_{i_j+1}$ for some N such that

(i) $L_0 \rightarrow^* N$ (*p7)

(ii) Either $H(\zeta) < H(\eta'[0, i_j+1])$ (*p8) or $i_j = i_l$, $H(\zeta) = H(\eta'[0, i_j+1])$ and ζ is ε-invariant.

Using this ζ, let $\delta = (\zeta; \eta'[i_j+1, m]) : N \leftrightarrow\!\!\!\rightarrow^* L_{i_j+1} \leftrightarrow\!\!\!\rightarrow^* L_m$. Then, by (*p8), we have

$H(\delta) \le H(\eta')$, (*p9)

Either δ is ε-invariant or $K_{ldis}(\delta) \ll_w K_{ldis}(\eta')$, (*p10)

since $H(\zeta) < H(\eta'[0, i_j+1])$ implies that $K_{ldis}(\delta) \ll_w K_{ldis}(\eta')$ by Property 5.

Let $\bar{\gamma}' = \gamma'[((L_0 \rightarrow^* N); \delta)/\eta]$. Note that $net(\eta) = \eta'$. Let $M' = M_0[u_1 \leftarrow N]$. Then, $\bar{\gamma}' : M_0 \leftrightarrow\!\!\!\rightarrow^* M_m$ $(= M_{n-1})$ is decomposed into two subsequences $\bar{\gamma}_1 : M_0 \leftrightarrow\!\!\!\rightarrow^* M'$ and $\bar{\gamma}_2 : M' \leftrightarrow\!\!\!\rightarrow^* M_m$. (Note that $L_0 = M_0/u_1$.) Since $M_0 \rightarrow^* M'$ by (*p7), it is sufficient to show that $P(k)$ holds for $(\bar{\gamma}_2; (M_m \leftrightarrow\!\!\!\rightarrow M_n)) : M' \leftrightarrow\!\!\!\rightarrow^* M_n$, instead of the original γ.

We can show that $weight_2(\bar{\gamma}_2) \ll_w weight_2(\gamma')$. Since $weight_2(\bar{\gamma}_2)$ is obtained from $weight_2(\gamma')$ by replacing $K_{ldis}(\eta')$ with $\bigsqcup_{v \in MR(\delta) \cap \bar{O}(\alpha/u_1)} K_{ldis}(net(\ \delta/v))$, if δ is ε-invariant, then obviously $weight_2(\bar{\gamma}_2) \ll_w weight_2(\gamma')$ holds by (*p9). Otherwise, $K_{ldis}(net(\delta)) \ll_w K_{ldis}(\eta')$ by (*p10) and Property 4, and thus $weight_2(\bar{\gamma}_2) \ll_w weight_2(\gamma')$ holds. By (*p6), $weight_2(\bar{\gamma}_2) \ll_w weight_2(\bar{\gamma})$. Hence, the induction hypothesis ensures that $P(k)$ holds for $(\bar{\gamma}_2; (M_{n-1} \leftrightarrow\!\!\!\rightarrow M_n))$. It follows that $P(k)$ also holds for γ. □

Proof of $Q(k)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\rightarrow M_n$ where $H(\gamma) \le k$, $M_0 = M$ and $M_n = N$. We prove $Q(k)$ by induction on $weight_3(\gamma)$ which is defined as follows:

$$weight_3(\gamma) = (H(\gamma), K_{width}(\gamma))$$

We use the lexicographic ordering $<$ as the ordering of $weight_3(\gamma)$'s and $\ll_w$ as the ordering of $K_{width}(\gamma)$'s. If $H(\gamma) \le k-1$ holds or γ is ε-invariant, then the proof can reduce to that of $Q(k-1)$. Note that if γ is ε-invariant, then we can apply the induction hypothesis to cut sequences of γ. Thus, assume that $H(\gamma) = k$ and γ has ε-reductions.

We first prove that if γ has no peaks, then $Q(k)$ holds. In this case, there exists l $(0 \le l \le n)$ such that for any ε-reduction $M_i \leftrightarrow\!\!\!\rightarrow M_{i+1}$ $(0 \le i < n)$ either $i < l$ and $M_i \overset{\varepsilon}{\rightarrow} M_{i+1}$ or $i \ge l$ and $M_i \overset{\varepsilon}{\leftarrow} M_{i+1}$ holds.

Therefore, let $\gamma_1 = \gamma[0, l] : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_l$ and $\gamma_2 = \gamma[l, n] : M_l \leftrightarrow\!\!\!\rightarrow M_{l+1} \cdots \leftrightarrow\!\!\!\rightarrow M_n$ where $\gamma = (\gamma_1; \gamma_2)$. Since γ has ε-reductions, we assume that γ_1 has ε-reductions. This does not lose generality, since otherwise we can consider γ^R instead of γ. (Note that $K_{width}(\gamma) = K_{width}(\gamma^R)$, so that $weight_3(\gamma) = weight_3(\gamma^R)$.) Then, by $P'(k)$, i $(0 \le i \le l)$ exists and

$$\eta : N \leftrightarrow\!\!\!\rightarrow^* M_i \text{ for some } N$$

such that $H(\eta) \le H(\gamma[0, i])$, $M_0 \rightarrow^* N$, and either (a) $H(\eta) < H(\gamma[0, i])$ or (b) $i = l$ and η is ε-invariant.

We first consider case (a): $H(\eta) < H(\gamma[0, i])$. Let $\gamma' = (\eta; \gamma[i, n]) : N \leftrightarrow\!\!\!\rightarrow^* M_i \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* M_n$.

By $H(\eta) < H(\gamma[0, i])$ and Property 5,

$$K_{width}(\gamma') \ll_w K_{width}(\gamma)$$

holds. Thus, by the induction hypothesis $Q(k)$ for γ', $\delta' : N \leftrightarrow\!\!\!\rightarrow^* L \leftrightarrow\!\!\!\rightarrow^* M_n$ exists for some L such that $N \rightarrow^* L$, $M_n \rightarrow^* L$ and $H(\delta') \le k$. Since $M_0 \rightarrow^* N$ exists and for $\gamma[0, i]; \eta^R : M_0 \leftrightarrow\!\!\!\rightarrow^* M_i \leftrightarrow\!\!\!\rightarrow^* N$, we have $H(\gamma[0, i]; \eta^R) \le k$, it follows that $Q(k)$ holds for γ.

There remains case (b), in which $i = l$ and $\eta : N \leftrightarrow\!\!\!\rightarrow^* M_l$ is ε-invariant. If γ_2 has no ε-reductions, then $\eta; \gamma_2$ is ε-invariant, and thus the induction hypothesis ensures that $Q(k)$ holds for $(\eta; \gamma_2)$. Hence, $Q(k)$ holds for γ, since $(\gamma[0, l]; \eta^R) : M_0 \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* N$ has a height at most k. Otherwise (i.e., if γ_2 has ε-reductions), we apply the same argument for γ_2^R as the above: by $P'(k)$, there exist j $(l \le j \le n)$ and

$$\bar{\eta} : N' \leftrightarrow\!\!\!\rightarrow^* M_j \text{ for some } N'$$

such that $H(\bar{\eta}) \le H(M_n \leftrightarrow\!\!\!\rightarrow M_{n-1} \leftrightarrow\!\!\!\rightarrow^* M_j)$, $M_n \rightarrow^* N'$ and either (a') $H(\bar{\eta}) < H(M_n \leftrightarrow\!\!\!\rightarrow M_{n-1} \cdots \leftrightarrow\!\!\!\rightarrow M_j)$ or (b') $j = l$ and

$\bar{\eta}$ is ε-invariant holds. If (a') holds, then $Q(k)$ holds for γ, by a similar argument to that in the above case (a).

Thus, there remains case (b'): $j = l$ and $\bar{\eta} : N' \leftrightarrow\!\!\!\rightarrow^* M_l$ is ε-invariant. In this case, let $\zeta = \eta; \bar{\eta}^R : N \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* N'$. Note also that $H(\zeta) \leq k$, $M_0 \to^* N$, ζ is ε-invariant and $M_n \to^* N'$. Note that for $\gamma_1; \eta^R : M_0 \leftrightarrow\!\!\!\rightarrow^* N$ and for $\bar{\eta}; \gamma_2 : N' \leftrightarrow\!\!\!\rightarrow^* M_n$, we have $H(\gamma_1; \eta^R) \leq k$ and $H(\bar{\eta}; \gamma_2) \leq k$.

Thus, the proof can be reduced to showing that $Q(k)$ holds for ζ. Since ζ is ε-invariant, the induction hypothesis $Q(k-1)$ ensures that $Q(k)$ holds for ζ. Hence, $Q(k)$ holds for γ.

We have proven that $Q(k)$ holds for γ without peaks. If γ has peaks, then when $S(n)$ is applied to γ, there exists a reduction sequence $\delta : M_0 \leftrightarrow\!\!\!\rightarrow^* M_n$ such that the following conditions (i) and (ii) hold:

(i) There are no peaks in δ.

(ii) $\delta \preceq \gamma$

By (ii) and Property 3(3), $weight_3(\delta) \leq weight_3(\gamma)$ holds. (Note that $K_{width}(\delta) \ll_s K_{width}(\gamma)$ implies that $K_{width}(\delta) \ll_w K_{width}(\gamma)$.) Thus, the proof for γ can be reduced to that for δ without peaks, which we have already given. □

We have proven that $Q(k)$ holds for all $k \geq 0$, and hence we have the following Theorem 1:

Theorem 1

All root-E-closed and strongly depth-preserving TRS's are CR. □

Theorem 1 obviously implies that non-E-overlapping and strongly depth-preserving TRS's are CR. The proof of the non-E-overlapping case derived from this proof of Theorem 1 is more refined than the old one in Ref. 11).

6. Weight-preserving TRS

By assigning a positive integer (called *weight*) to each function symbol, we can naturally extend the notion of depth to that of weight, and obtain a similar result to Theorem 1 for strongly weight-preserving TRS's.

Definition of
⟨strongly weight-preserving TRS R⟩

For a weight-assigning function $w : F \to \{1, 2, 3, \cdots\}$, let $W_w(u, M)$ be the total of the weights of function symbols occurring from the root to occurrence u on term M. Formally, $W_w(\varepsilon, x) = 0$, $W_w(\varepsilon, fM_1 \cdots M_n) = w(f)$, $W_w(iu, fM_1 \cdots M_n) = w(f) + W_w(u, M_i)$ where $x \in X$, $f \in F$ and $M_i \in T$ $(1 \leq i \leq n)$. A TRS R is *strongly w-weight-preserving* if $\forall \alpha \to \beta \in R$ $\forall x \in V(\beta)$ $Max\{W_w(v, \beta) \mid v \in O_x(\beta)\} \leq Min\{W_w(u, \alpha) \mid u \in O_x(\alpha)\}$.

A TRS R is *strongly weight-preserving* if $\exists w : F \to \{1, 2, 3, \cdots\}$ R is strongly w-weight-preserving.

Example 2. $R_4 = \{f(x, x) \to a,\ c \to h(c, g(c)),\ h(g_3(x), g_1(g_2(x))) \to f(x, h(x, g(x)))\}$ where $x \in X$, $a, c, f, h, g, g_1, g_2, g_3 \in F$. R_4 is strongly w-weight-preserving for a weight-assigning function w such that $w(g_3) = 2$ and $w(k) = 1$ for all $k \in F - \{g_3\}$. (Note that $W_w(11, h(g_3(x), g_1(g_2(x)))) = 3$.) But R_4 is not strongly depth-preserving.

The problem of deciding for a given TRS whether it is strongly weight-preserving or not can be reduced to that of solving integer programming.

Example 3. For a TRS R_4, we have the following integer programming problem:

$$k \geq 1$$
$$h + g_3 \geq f + h + g$$
$$h + g_1 + g_2 \geq f + h + g$$

where for all $k \in K = \{a, c, f, g, h, g_1, g_2, g_3\}$. These inequalities hold for a weight-assigning function w such that $w(g_3) = 2$ and $w(k) = 1$ for all $k \in K - \{g_3\}$. Thus, R_4 is strongly w-weight-preserving.

If TRS R is strongly depth-preserving, then obviously R is strongly weight-preserving, since R is strongly w_1-weight-preserving for the weight-assigning function w_1 such that $w_1(f) = 1$ for all $f \in F$.

For any strongly w-weight-preserving TRS R, we can construct a strongly depth-preserving TRS $\bar{R}_w$ that can simulate reductions of R. For this purpose, we define a set of new function symbols $\bar{F}$ and a translation $\psi : F \to \bar{F}^*$ as follows:

$$\bar{F} = \{f_1, f_2, \cdots, f_k \mid f \in F, w(f) = k\}$$

where $\mathsf{a}(f_i) = 1,\ 1 \leq i < k$ and $\mathsf{a}(f_k) = \mathsf{a}(f)$

$$\psi(f) = f_1 \cdot f_2 \cdots f_k \text{ for } f \in F \text{ with } w(f) = k$$

Here, $(f_1 \cdot f_2 \cdot \cdots \cdot f_k)M_1 \cdots M_n = f_1(f_2 \cdots (f_k M_1 \cdots M_n) \cdots)$ for $M_1, \cdots, M_n \in T$, where $\mathsf{a}(f_k) = n$.

Translation ψ is extended to the translation: $T \to \bar{T}^*$ as follows:

$$\psi(x) = x \text{ for } x \in X$$
$$\psi(fM_1 \cdots M_n) = \psi(f)\psi(M_1) \cdots \psi(M_n)$$

for $f \in F$, $M_1, \cdots, M_n \in T$. Here, $\bar{T}$ is the set of terms constructed from X and $\bar{F}$.

Using this translation ψ, we define a new TRS $\bar{R}_w$ by

$$\bar{R}_w = \{\psi(\alpha) \to \psi(\beta) \mid \alpha \to \beta \in R\}$$

It can be proven that R is strongly w-weight-preserving iff $\bar{R}_w$ is strongly depth-preserving.

A strongly w-weight preserving TRS R is said to be root-E-closed if the corresponding $\bar{R}_w$ is root-E-closed. Hence, by Theorem 1, we have the following result:

Theorem 2

All strongly weight-preserving and root-E-closed TRS's are CR. □

Note. We can give a more direct definition of root-E-closedness for TRS R using the maximal weights of terms instead of the heights (maximal depths) of terms, that is, let $W_w(\gamma) = Max\{W_w(M_i) \mid 0 \leq i \leq n\}$ where γ is a reduction sequence such that $\gamma : M_0 \leftrightarrow\!\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\!\rightarrow \cdots M_n$, and we use $W_w(M)$ instead of $H(M)$ and $W_w(\gamma)$ instead of $H(\gamma)$. And $left(\gamma, h)$, $right(\gamma, h)$, $ldis(\gamma, h)$ and $width(\gamma, h)$ are redefined by using $W_w(M)$ instead of $H(\gamma)$. Also, $K_{ldis}(\gamma)$, $K_{right}(\gamma)$, $K_{width}(\gamma)$ and $\delta \preceq \gamma$ are redefined by the corresponding new definitions.

Strongly w-weight preserving TRS R is said to be root-E-closed if R is root-E-closed under the above new definitions. Then, we can easily obtain the same result as Theorem 2 using the new version of root-E-closedness. Since non-E-overlapping TRS's are obviously root-E-closed, we have the following corollary:

Corollary 1

All non-E-overlapping and strongly weight-preserving TRS's are CR. □

Example 4. Since R_4 is non-E-overlapping and strongly weight-preserving, Corollary 1 ensures that R_4 is CR. TRS $R = \{\ f(x) \to g(h(x), x),\ g(x, x) \to a,\ b \to h(b)\}$[12] in Section 1 is non-E-overlapping and strongly w-weight-preserving for a weight-assigning function w such that $w(f) = 2$ and $w(k) = 1$ for $k \in \{a, b, g, h\}$, and thus R is CR.

7. Sufficient Condition

In Section 5, we showed that all root-E-closed and strongly depth-preserving TRS's are CR. Unfortunately, it is undecidable for any strongly depth-preserving TRS R whether R is root-E-closed, whereas it is obviously decidable for any TRS R whether R is strongly depth-preserving. Thus, we need to obtain decidable sufficient conditions for ensuring root-E-closedness. In this section, we give such a condition.

We first give the definition of strongly root-overlapping TRS's which are closely related to root-E-overlapping TRS's. We show that if TRS R is strongly root-overlapping and satisfies the condition $(*)$ of root-E-closedness, then TRS R is root-E-closed. (Note that TRS R is root-E-closed iff R is root-E-overlapping and satisfies the condition $(*)$ of root-E-closedness.)

Definition. For a term α, let $\bar{\alpha}$ be a linearization of α, that is, a linear term satisfying the condition that $\theta(\bar{\alpha}) = \alpha$ for some substitution θ such that $\theta(x) \in X$ for all $x \in X$, (e.g., $\bar{\alpha} = f(x, y)$ is a linearization of $\alpha = f(x, x)$ where $f \in F$ and $x, y \in X$). A pair of rewrite rules $\alpha \to \beta$ and $\alpha' \to \beta'$ is *strongly overlapping (at u)* if the pair of $\bar{\alpha} \to \beta$ and $\bar{\alpha}' \to \beta'$ is overlapping at u (i.e., $u \in \bar{O}(\bar{\alpha})$ and $\sigma(\bar{\alpha}/u) = \sigma'(\bar{\alpha}')$ for some substitutions σ and σ'.). If $u = \varepsilon$, then the pair is *strongly root-overlapping.* TRS R is strongly root-overlapping if all strongly overlapping pairs of rewrite rules are strongly root-overlapping.

Lemma 1

If TRS R is strongly root-overlapping and satisfies the condition $(*)$ of root-E-closedness, then R is root-E-overlapping or non-E-overlapping.

Proof

To prove this, we assume to the contrary that R is E-overlapping and not root-E-overlapping. Let $\gamma_0 : \sigma(\alpha)[u \leftarrow \sigma'(\beta')] \leftarrow \sigma(\alpha)[u \leftarrow \sigma'(\alpha')] \overset{>u}{\leftrightarrow\!\!\!\!\rightarrow^*} \sigma(\alpha) \to \sigma(\beta)$ be an E-overlapping but non-root-E-overlapping sequence with the minimal number of parallel steps where $\alpha \to \beta$, $\alpha' \to \beta' \in R$ and $u \in \bar{O}(\alpha) - \{\varepsilon\}$. Let $\gamma : \sigma'(\alpha') \overset{\varepsilon-inv}{\leftrightarrow\!\!\!\!\rightarrow^*} \sigma(\alpha/u)$ be the cut subsequence of γ_0, and if there is more than one sequence with the minimal number of parallel steps, then we select the sequence γ_0 whose subsequence γ has the minimal weight of $weight_4$ where $weight_4(\gamma) = \{H(\gamma/v) \mid v \in MR(\gamma) \cap \bar{O}(\alpha/u) \cap \bar{O}(\alpha')\}_m$. Here, we use the multiset ordering $\ll$ as the ordering of $weight_4(\gamma)$'s. Note that if δ is E-overlapping and $|\delta|_p < |\gamma_0|_p$, then δ is root-E-overlapping. We show that $weight_4(\gamma) = \phi$.

To the contrary, we assume that $weight_4(\gamma) \neq \phi$. Let $v \in MR(\gamma) \cap \bar{O}(\alpha/u) \cap \bar{O}(\alpha')$. Note that γ/v has ε-reductions. Therefore, if $\gamma/v : \sigma'(\alpha'/v) \leftrightarrow\!\!\!\!\rightarrow^* \sigma(\alpha/uv)$ had no peaks, then from γ/v we could obtain an E-overlapping but non-root-E-overlapping sequence which contradicts the minimality of γ_0. Thus, γ/v must have peaks. Therefore, we apply $S(|\gamma/v|_p)^\star$ and obtain a parallel reduction sequence $\delta_v : \sigma(\alpha/uv)$

$\leftrightarrow\!\!\!\!\leftrightarrow^* \ \sigma'(\alpha'/v)$ that has no peaks and $|\delta_v|_p \leq |\gamma/v|_p$. If δ_v has an ε-reduction, then δ_v again contradicts the minimality of γ. Otherwise, let $\gamma' = \gamma[\delta_v/(\gamma/v)]$; then $weight_4(\gamma') \ll weight_4(\gamma)$ holds, which again contradicts the minimality of $weight_4$ of γ.

Thus, $weight_4(\gamma) = \phi$, and thus γ is standard. Therefore, the pair of $\alpha \to \beta$ and $\alpha' \to \beta'$ is strongly overlapping at $u(\neq \varepsilon)$. However, R is strongly root-overlapping, which is a contradiction. Hence, R is root-E-overlapping or non-E-overlapping. □

Note that whether TRS R is strongly root-overlapping is decidable. Next, we consider how to check the condition $(*)$ of root-E-closedness. This approach is analogous to that of Ref. 6).

Condition I

For any strongly root-overlapping pair $(\alpha \to \beta,\ \alpha' \to \beta')$ where $V(\alpha) \cap V(\alpha') = \phi$, there exists a parallel reduction sequence either $\beta \dashrightarrow\!\!\!\!\to_{R'} M \leftrightarrow\!\!\!\!\leftrightarrow \beta'$ for some M where $V(M) \cap V(\beta) = \phi$ and $H(\beta) \leq Max(H(\alpha'), H(\beta'))$, or $\beta \leftrightarrow\!\!\!\!\leftrightarrow N \dashrightarrow\!\!\!\!\to_{R'} \beta'$ for some N where $V(N) \cap V(\beta') = \phi$ and $H(\beta') \leq Max(H(\alpha), H(\beta))$. Here, $\leftrightarrow\!\!\!\!\leftrightarrow$ is a one-step parallel reduction of the original TRS R and $\dashrightarrow\!\!\!\!\to_{R'}$ is a parallel reduction of the TRS $R' = \{\alpha/u \to \alpha'/u \mid u \in Min\{O_X(\alpha) \cup O_X(\alpha')\},\ V(\alpha/u) \neq \phi \text{ and } V(\alpha'/u) \neq \phi\}$ where any variable appearing in rewrite rules in TRS R' is regarded as a constant and no substitution for the variable is allowed in rewriting $\dashrightarrow\!\!\!\!\to_{R'}$. Note that it is decidable whether TRS R satisfies Condition I.

Lemma 2

Let TRS R be strongly depth-preserving. If R satisfies Condition I, then R satisfies the condition $(*)$ of root-E-closedness.

Proof

Let γ be any standard root-E-overlapping sequence $\gamma : M_0 \leftrightarrow\!\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\!\leftrightarrow M_{n-1} \leftrightarrow\!\!\!\!\leftrightarrow M_n$ with $|\gamma|_p = |\gamma|_{np}$ where $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$, $M_{n-1} = \sigma'(\alpha')$, $M_n = \sigma'(\beta')$ for some $\alpha \to \beta,\ \alpha' \to \beta' \in R$ and $\sigma, \sigma' : X \to T$. Without loss of generality, we can assume that $V(\alpha) \cap V(\alpha') = \phi$ and $\sigma = \sigma'$. Since the pair $(\alpha \to \beta,\ \alpha' \to \beta')$ is strongly root-overlapping, by Condition I either $\xi : \beta \dashrightarrow\!\!\!\!\to_{R'} M \leftrightarrow\!\!\!\!\leftrightarrow \beta'$ or $\xi' : \beta \leftrightarrow\!\!\!\!\leftrightarrow N \dashrightarrow\!\!\!\!\to_{R'} \beta'$.

We consider only the former case. (The proof of the latter case is similar.) Let $\gamma' = \gamma[1, n-1]$. For each $u \in Min(O_X(\alpha) \cup O_X(\alpha'))$, we have the cut sequence $\gamma'/u : \sigma(\alpha/u) \leftrightarrow\!\!\!\!\leftrightarrow^* \ \sigma'(\alpha'/u)$, so that by associating γ'/u with the rule $\alpha/u \to \alpha'/u$ in R', we obtain a parallel reduction sequence $\delta : \sigma(\beta) \leftrightarrow\!\!\!\!\leftrightarrow^* \ \sigma'(M)$ from $\beta \dashrightarrow\!\!\!\!\to_{R'} M$. Note that $|\delta|_p = |\gamma'|_p = |\gamma|_p - 2$.

Using the subsequence $M \leftrightarrow\!\!\!\!\leftrightarrow \beta'$ of ξ, let $\delta' = (\sigma(\beta) \leftrightarrow\!\!\!\!\leftrightarrow \sigma(\beta); \delta; \sigma'(M) \leftrightarrow\!\!\!\!\leftrightarrow \sigma'(\beta')) : N_0\ (= M_0) \leftrightarrow\!\!\!\!\leftrightarrow N_1\ (= M_0) \leftrightarrow\!\!\!\!\leftrightarrow N_2 \cdots \leftrightarrow\!\!\!\!\leftrightarrow N_{n-1}\ (= \sigma'(M)) \leftrightarrow\!\!\!\!\leftrightarrow N_n\ (= M_n)$. Note that

$$|\delta'|_p = |\gamma|_p \text{ and } |\delta'|_{np} < |\gamma|_{np}.$$

Let $v \in R(\beta \dashrightarrow\!\!\!\!\to_{R'} M)$. Then there exists some u such that $\beta/v = \alpha/u$ and $\alpha/u \to \alpha'/u \in R'$. Note that $\gamma'/u : \sigma(\alpha/u) \leftrightarrow\!\!\!\!\leftrightarrow^* \ \sigma'(\alpha'/u)$. Since β/v contains at least one variable by the definition of R', then $|v| \leq |u|$ holds by the strongly depth-preserving property of TRS R, and thus we have $|v| + H(\delta/v) \leq |u| + H(\gamma'/u) \leq H(\gamma)$. Thus, for any i $(0 \leq i \leq n)$, $H(N_i) \leq Max(H(\beta), H(M_i))$ holds, since for every $t \in O_X(\beta)$ there exists some $v \in R(\beta \dashrightarrow\!\!\!\!\to_{R'} M)$ such that $v \leq t$ by $V(\beta) \cap V(M) = \phi$, as required by Condition I. If $H(\beta) > H(M_i)$, then $H(M_0)(= H(\sigma(\beta))) \geq H(N_i)$ holds. Since $H(\beta) \leq Max(H(\alpha'), H(\beta'))$ by Condition I, we have $H(N_i) \leq Max(H(M_{n-1}), H(M_n))$. Thus, in either case $\exists j, j'$ such that $j \leq i \wedge H(N_i) \leq H(M_j)$, and $i \leq j' \wedge H(N_i) \leq H(M_{j'})$ for every i $(0 \leq i \leq n)$.

Hence, by Property 7, $K_{ldis}(\delta') \lesseqqgtr_s K_{ldis}(\gamma)$ and $K_{right}(\delta') \lesseqqgtr_s K_{right}(\gamma)$, and thus $\delta' \preceq \gamma$ and $|\delta'|_{np} < |\gamma|_{np}$ hold. That is, the condition $(*)$ of root-E-closedness holds. □

By Lemmata 1 and 2 and Theorem 1, we have the following theorem which gives a decidable sufficient condition for root-E-closedness:

Theorem 3

Let TRS R be strongly depth-preserving. If R is strongly root-overlapping and satisfies Condition I, then R is root-E-closed or non-E-overlapping; Therefore, R is CR.

Proof

By Lemma 2, R satisfies the condition $(*)$ of root-E-closedness, and thus by Lemma 1, R is root-E-overlapping or non-E-overlapping. If R is non-E-overlapping, then R is CR[11]. Otherwise (i.e.,if R is root-E-overlapping), R is root-E-closed, and thus R is CR by Theorem 1. □

Example 5. To show that R_3 in Example 1 of Section 3 is CR, we show that R_3 is root-E-closed. It is obvious that R_3

☆ Let $l = |\gamma_0|_p$, that is, the minimal number of parallel steps of non-root-E-overlapping sequences. If every root-E-overlapping sequence with $(l-1)$ parallel steps or less satisfies (i) and (ii) of the condition $(*)$ of root-E-closedness, then we can prove that $S(n)$ holds for any $n < l$. The proof is similar to that of $S(n)$ in Section 5, and is therefore omitted.

is strongly root-overlapping. For a strongly root-overlapping pair $(f(x,x) \rightarrow h(x,x), f(g(x'),x') \rightarrow a)$, we have $R' = \{x \rightarrow g(x'), x \rightarrow x'\}$ and a parallel reduction sequence $h(x,x) \mathrel{+\!\!\!+\!\!\!\rightarrow}_{R'} h(g(x'),x') \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!\rightarrow} a$, and $H(h(x,x)) \leq Max(H(f(g(x'),x')), H(a))$. For a strongly root-overlapping pair $(f(g(x'),x') \rightarrow a, f(x,x) \rightarrow h(x,x))$, we have $R'' = \{g'(x) \rightarrow x, x' \rightarrow x\}$ and $a \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!\rightarrow} h(g(x'),x') \mathrel{+\!\!\!+\!\!\!\rightarrow}_{R''} h(x,x)$, and $H(h(x,x)) \leq Max(H(f(g(x'),x')), H(a))$. Thus, R_3 satisfies Condition I, and hence R_3 is root-E-closed; Therefore, R_3 is CR.

8. Conclusion

In this paper, we have shown that all root-E-closed and strongly depth-preserving TRS's are CR, and given a decidable sufficient condition that ensures the root-E-closed condition. Note that root-E-closed TRS R is root-E-overlapping (i.e., every E-overlapping sequence is $\sigma(\beta) \leftarrow \sigma(\alpha) \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!\rightarrow}^* \sigma'(\alpha') \rightarrow \sigma'(\beta')$ for some $\alpha \rightarrow \beta$, $\alpha' \rightarrow \beta' \in R$ and mappings σ, σ'). So, it will be the next step will therefore be to find sufficient conditions for the CR property of E-overlapping (i.e., non-root-E-overlapping) and strongly depth-preserving TRS's.

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References

1) Dershowitz, N. and Jouannaud, J.-P.: Rewrite Systems, van Leeuwen, J. (Ed.), *Handbook of Theoretical Computer Science*, Vol.B, pp.243–320, North-Holland, Amsterdam (1990).
2) Huet, G.: Confluent reductions:abstract properties and applications to term rewrite systems, *J. ACM*, Vol.27, No.4, pp.797–821 (1980).
3) Matsuura, K., Oyamaguchi, M., Ohta, Y. and Ogawa, M.: On the E-overlapping Property of NonLinear Term Rewriting Systems, *Trans. IEICE Japan*, Vol.J80-D-1, No.11, pp.847–855 (1997).
4) Ogawa, M. and Ono, S.: On the uniquely converging property of nonlinear term rewriting systems, Technical Report of IEICE, COMP 89-7, pp.61–70 (1989).
5) Oyamaguchi, M. and Ohta, Y.: On the Confluent Property of Right-Ground Term Rewriting Systems, *Trans. IEICE Japan*, Vol.J76-D-I, No.2, pp.39–45 (1993).
6) Toyama, Y. and Oyamaguchi, M.: Church-Rosser Property and Unique Normal Form Property of Non-Duplicating Term Rewriting Systems, *4th International Workshop on Conditional TRS*, LNCS, Vol.968, pp.316–331, Springer (1994).
7) Ohta, Y. Oyamaguchi, M. and Toyama,Y.: On the Church-Rosser Property of Simple-Right-Linear TRS's, *Trans. IEICE Japan*, Vol.J78-D-I, No.3, pp.263–268 (1995).
8) Oyamaguchi, M. and Toyama, Y.: On the Church-Rosser property of E-overlapping and simple-right-linear TRS's, Technical Report of IEICE, COMP 94-29, pp.47–56 (1994).
9) Oyamaguchi, M. and Gomi, H.: Some Results on the CR property of Non-E-overlapping and Depth-Preserving TRS's, RIMS Kokyuroku 918, pp.150–159 (1995).
10) Gomi, H., Oyamaguchi,M. and Ohta,Y.: On the Church-Rosser Property of Non-E-overlapping and Weight-Preserving TRS's, RIMS Kokyuroku 950, pp.167–173 (1996).
11) Gomi, H., Oyamaguchi, M. and Ohta, Y.: On the Church-Rosser Property of Non-E-overlapping and Strongly Depth-Preserving Term Rewriting Systems, *Trans. IPS. Japan*, Vol.37, No.12, pp.2147–2160 (1996).
12) Middeldorp, A., Okui, S. and Ida, T.: Lazy Narrowing: Strong Completeness and Eager Variable Elimination, *Theoretical Computer Science*, Vol.167, No.1, 2, pp.95–130 (1996).

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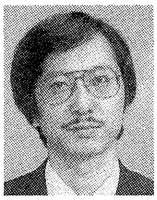

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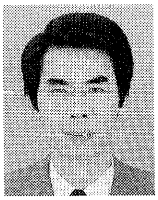

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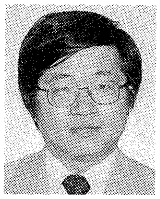

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