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Technical Note

A Note on the Circuit-switched Fixed Routing in Networks

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This note considers the permutation routing problem on circuit-switched fixed routing networks. It is known that the size of optimal scheduling for any permutation on a 2-dimensional square mesh with N vertices is $O(\sqrt{N})$. In this note, we show a scheduling for any permutation with optimal size of $O(\sqrt{N}/\log N)$ for the N-vertex hypercube and with optimal size of $O(\sqrt{N}/d)$ for the N-vertex d-dimensional square mesh and torus. We also show that such a scheduling can be found in polynomial time by a unified approach.

1. Circuit-switched Fixed Routing

The circuit-switched fixed-routing model has been adopted for some parallel computer systems, such as iPSC-2 and iPSC-860 by Intel, NCUBE/10 by nCUBE, and Symult 2010 by Ametek[6]. In this model, a fixed path is dedicated to every source-destination pair and data is pipelined through the path. Once a fixed path is established for a source-destination pair, the path exclusively uses all the edges that it traverses and no other fixed path that uses one of those same edges can be established simultaneously. Therefore, if multiple source-destination pairs wish to communicate simultaneously, the fixed paths dedicated to those source-destination pairs must be edge-disjoint.

Let G be a graph representing a network, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A routing ρ on G is a mapping from the set of all ordered pairs of vertices in G to the set of all paths in G such that $\rho([u,v])$ is a path connecting u and v. A communication request on G is a set of ordered pairs of vertices in G. If $[u,v]$ is in a communication request, u is called the source and v the destination of the pair. A communication request on G is called a partial permutation if each vertex appears in the request at most once as a source and at most once as a destination. A permutation is a partial permutation with $|V(G)|$ source-destination pairs.

Let G be a graph with routing ρ. For a source-destination pair $[u,v]$, $\rho([u,v])$ is called a fixed path dedicated to $[u,v]$. A scheduling for a permutation π is a decomposition of π into partial permutations such that the fixed paths dedicated to the source-destination pairs in each partial permutation are edge-disjoint. The size of a scheduling is the number of partial permutations in the decomposition. Let $\sigma(\pi,\rho,G)$ be the minimum size of a scheduling for a permutation π on a graph G with routing ρ. Define that

$$\sigma(\rho,G) = \max_{\pi} \sigma(\pi,\rho,G), \text{ and}$$
$$\sigma(G) = \min_{\rho} \sigma(\rho,G).$$

Since the impact of vertex conflict and path length is negligible in circuit-switched fixed-routing networks as mentioned by Bokhari[1], $\sigma(G)$ is the dominant factor for the communication overhead in circuit-switched fixed-routing network G. Therefore, designing a routing ρ that attains $\sigma(G)$ and finding a scheduling with size $\sigma(\rho,G)$ are fundamental problems to minimize the communication overhead when realizing a permutation on a circuit-switched fixed-routing network G. The problems were first considered by Youssef[6]. Among other results, it is shown in Ref. 6) that $\sigma(G) = O(\sqrt{N})$ if G is a 2-dimensional square mesh with N vertices.

2. Upper Bound for Product Graphs

The mesh is a typical example of product graphs, many of which have emerged as attractive interconnection graphs for large multiprocessor systems. The product of two graphs G and H, denoted by $G \times H$, is the graph defined as follows:

$$V(G \times H) = V(G) \times V(H);$$
$$E(G \times H) = \{([u,v],[u',v]) \mid (u,u') \in E(G)\} \cup \{([u,v],[u,v']) \mid (v,v') \in E(H)\}.$$

We show the following upper bound for product graphs.

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Theorem 1 Let G_1 and G_2 be N_1- and N_2-vertex graphs with p_1 and p_2 edge-disjoint spanning trees, respectively. Then,

$$\sigma(G_1 \times G_2) \le \max\{\lceil N_1/p_1\rceil, \lceil N_2/p_2\rceil\}.$$

Proof: We prove the theorem by a series of lemmas. Let $T_{1,0}$, $T_{1,1}, \ldots$, and T_{1,p_1-1} denote p_1 edge-disjoint spanning trees of G_1 and $T_{2,0}$, $T_{2,1}, \ldots$, and T_{2,p_2-1} denote p_2 edge-disjoint spanning trees of G_2.

Assume without loss of generality that $V(G_1) = [N_1]$ and $V(G_2) = [N_2]$, where $[N] = \{0, 1, \ldots, N-1\}$ for any positive integer N. A graph with just one vertex x is denoted by x. For any two vertices $u = [u_1, u_2]$ and $v = [v_1, v_2]$ in $G_1 \times G_2$, $\rho([u, v])$ is defined as the concatenation of the unique path from u to $[u_1, v_2]$ in $u_1 \times T_{2,u_2 \bmod p_2}$ and the unique path from $[u_1, v_2]$ to v in $T_{1,v_1 \bmod p_1} \times v_2$.

Lemma 1 Let $u = [u_1, u_2]$, $v = [v_1, v_2]$, $u' = [u'_1, u'_2]$, $v' = [v'_1, v'_2] \in V(G_1 \times G_2)$. If $[u_1, u_2 \bmod p_2] \ne [u'_1, u'_2 \bmod p_2]$ and $[v_1 \bmod p_1, v_2] \ne [v'_1 \bmod p_1, v'_2]$ then $\rho([u, v])$ and $\rho([u', v'])$ are edge-disjoint.

Proof of Lemma 1: The lemma follows from the following two facts: (i) If $u_1 \ne u'_1$ $[v_2 \ne v'_2]$ or $u_2 \bmod p_2 \ne u'_2 \bmod p_2$ $[v_1 \bmod p_1 \ne v'_1 \bmod p_1]$ then $u_1 \times T_{2,u_2 \bmod p_2}$ $[T_{1,v_1 \bmod p_1} \times v_2]$ and $u'_1 \times T_{2,u'_2 \bmod p_2}$ $[T_{1,v'_1 \bmod p_1} \times v'_2]$ are edge-disjoint; (ii) $u_1 \times T_{2,u_2 \bmod p_2}$ $[u'_1 \times T_{2,u'_2 \bmod p_2}]$ and $T_{1,v'_1 \bmod p_1} \times v'_2$ $[T_{1,v_1 \bmod p_1} \times v_2]$ are edge-disjoint. □

For any permutation π on $G_1 \times G_2$, a bipartite multigraph B_π with bipartition $(X(B_\pi), Y(B_\pi))$ is defined as follows:

$$X(B_\pi) = \{r_{[x,i]} \mid x \in [N_1],\ i \in [p_2]\};$$
$$Y(B_\pi) = \{c_{[j,y]} \mid j \in [p_1],\ y \in [N_2]\};$$

Any two vertices $r_{[x,i]} \in X(B_\pi)$ and $c_{[j,y]} \in Y(B_\pi)$ are joined by

$$\left|\left\{ u \;\middle|\; \begin{array}{l} u \in V(G_1 \times G_2) \\ u = [u_1, u_2], \\ \pi(u) = [v_1, v_2], \\ [u_1, u_2 \bmod p_2] = [x, i], \\ [v_1 \bmod p_1, v_2] = [j, y] \end{array} \right\}\right|$$

parallel edges.

Notice that there is a one-to-one correspondence between the source-destination pairs of π and the edges in B_π. Let e_u denote the edge in B_π corresponding to $[u, \pi(u)]$.

For any graph G, let $\deg_G(v)$ denote the degree of a vertex v in G and let $\Delta(G)$ denote the maximum degree of a vertex in G.

Lemma 2

$$\Delta(B_\pi) \le \max\{\lceil N_1/p_1\rceil, \lceil N_2/p_2\rceil\}.$$

Proof of Lemma 2: Consider any $r_{[x,i]} \in X(B_\pi)$. Since

$$\left|\left\{ u \;\middle|\; \begin{array}{l} u \in V(G_1 \times G_2), \\ u = [u_1, u_2], \\ [u_1, u_2 \bmod p_2] = [x, i] \end{array} \right\}\right| \le \left\lceil \frac{N_2}{p_2} \right\rceil$$

we have

$$\deg_{B_\pi}(r_{[x,i]}) \le \left\lceil \frac{N_2}{p_2} \right\rceil.$$

Similarly, for any $c_{[j,y]} \in Y(B_\pi)$,

$$\left|\left\{ u \;\middle|\; \begin{array}{l} u \in V(G_1 \times G_2), \\ \pi(u) = [v_1, v_2], \\ [v_1 \bmod p_1, v_2] = [j, y] \end{array} \right\}\right| \le \left\lceil \frac{N_1}{p_1} \right\rceil$$

and so we have

$$\deg_{B_\pi}(c_{[j,y]}) \le \left\lceil \frac{N_1}{p_1} \right\rceil.$$

Hence $\Delta(B_\pi) \le \max\{\lceil N_1/p_1\rceil, \lceil N_2/p_2\rceil\}$. □

For any graph G, a mapping $f : E(G) \to \mathbb{N}$ is called an edge-coloring of G if $f(e_1) \ne f(e_2)$ for any two adjacent edges e_1 and e_2 in G, where $\mathbb{N}$ denotes the set of natural numbers. An edge-coloring f of G is called a k-edge-coloring of G if $f(E(G)) \subseteq [k]$. It is well-known that, for any bipartite multigraph G, there exists $\Delta(G)$-edge-coloring of G. In fact, the following lemma was proved by Cole and Hopcroft in Ref. 2).

Lemma 3[2)] Let G be a bipartite multigraph. Then we can find a $\Delta(G)$-edge-coloring of G in $O(|E(G)| \log |E(G)|)$ time. □

By Lemma 3, there exists a $\Delta(B_\pi)$-edge-coloring f of B_π. For any $i \in [\Delta(B_\pi)]$, define

$$\pi_i = \left\{ [u, \pi(u)] \;\middle|\; \begin{array}{l} u \in V(G_1 \times G_2), \\ f(e_u) = i \end{array} \right\}.$$

Let $u = [u_1, u_2]$ and $u' = [u'_1, u'_2]$ be any distinct vertices of $G_1 \times G_2$ such that $[u, \pi(u)], [u', \pi(u')] \in \pi_i$. Since $f(e_u) = f(e_{u'})$, we have

$$[u_1, u_2 \bmod p_2] \ne [u'_1, u'_2 \bmod p_2], \text{ and}$$
$$[v_1 \bmod p_1, v_2] \ne [v'_1 \bmod p_1, v'_2],$$

where $\pi(u) = [v_1, v_2]$, $\pi(u') = [v'_1, v'_2]$. Thus, by Lemma 1, $\rho([u, \pi(u)])$ and $\rho([u', \pi(u')])$ are edge-disjoint, and hence $(\pi_0, \ldots, \pi_{\Delta(B_\pi)-1})$ is a scheduling for π. Since $\Delta(B_\pi) \le \max\{\lceil N_1/p_1\rceil, \lceil N_2/p_2\rceil\}$ by Lemma 2, we have $\sigma(G_1 \times G_2) \le \max\{\lceil N_1/p_1\rceil, \lceil N_2/p_2\rceil\}$. ∎

Since $|E(B_\pi)| = N$ by the definition of B_π, we can find a $\Delta(B_\pi)$-edge-coloring of B_π in $O(N \log N)$ time by Lemma 3. Hence we can find the scheduling for π in $O(N \log N)$ time.

3. General Lower Bound

We have the following general lower bound for $\sigma(G)$.

Theorem 2 For any N-vertex graph G,

$$\sigma(G) = \Omega(\sqrt{N}/\Delta(G)),$$

where $\Delta(G)$ is the maximum vertex degree of G. ∎

Kaklamanis, et al.[4] showed that for any N-vertex graph G and any packet-switched oblivious routing ρ on G, there exists a permutation π such that ρ requires $\Omega(\sqrt{N}/\Delta(G))$ steps to realize π. Since the lower bound is derived from an estimate of the edge congestion, the same lower bound can be derived for the circuit-switched fixed routing by a slight modification of argument.

4. Tight Bounds for Some Product Graphs

From the theorems above, we can derive tight bounds for some product graphs. We denote the N-vertex path and cycle by P_N and C_N, respectively, and let $\prod_{i=1}^{d} G_i = G_1 \times G_2 \times \cdots \times G_d$.

$$R_d(k) = \prod_{i=1}^{d} P_k$$

is the d-dimensional k-sided mesh,

$$D_d(k) = \prod_{i=1}^{d} C_k$$

is the d-dimensional k-sided torus, and

$$Q_n = \prod_{i=1}^{n} P_2$$

is the n-dimensional cube.

Theorem 3

$$\sigma(Q_n) = \Theta(\sqrt{N}/\log N)$$

where $N = |V(Q_n)| = 2^n$;

$$\sigma(D_d(k)) = \Theta(\sqrt{N}/d)$$

if d is even where $N = |V(D_d(k))| = k^d$;

$$\sigma(R_d(k)) = \Theta(\sqrt{N}/d)$$

if d is even where $N = |V(R_d(k))| = k^d$.

Proof: The lower bounds can be derived from Theorem 2 and the fact that $\Delta(Q_n) = n$ and $\Delta(D_d(k)) = \Delta(R_d(k)) = 2d$.

The upper bounds can be derived from Theorem 1 as follows. We first observe that

$$Q_n = Q_{\lceil n/2 \rceil} \times Q_{\lfloor n/2 \rfloor},$$
$$D_d(k) = D_{d/2}(k) \times D_{d/2}(k), \text{ and}$$
$$R_d(k) = R_{d/2}(k) \times R_{d/2}(k).$$

We also observe that Q_n is n-edge-connected, $D_d(k)$ is $2d$-edge-connected, and $R_d(k)$ is d-edge-connected. Since it is well-known that an m-edge-connected graph has $\lceil (m-1)/2 \rceil$ edge-disjoint spanning trees[3),5)], we have the desired upper bounds. ∎

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