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Modularity of Simple Termination of Term Rewriting Systems †

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1 Introduction

A term rewriting system (TRS) R is a program expressed as a finite set of rewrite rules. The *direct sum* $R_0 \oplus R_1$ of two term rewriting systems R_0 and R_1 is defined only when the set of function symbols contained in R_0 and R_1 is disjoint; then $R_0 \oplus R_1$ is just the union of the both set of rules. A property of term rewriting systems is *modular* if, $R_0 \oplus R_1$ has the property iff both R_0 and R_1 have the same property. A TRS is said to be *terminating* if, for any initial term, there is no infinite rewrite sequence.

For a long time, the modularity of termination had been conjectured, but it was refuted by the counterexample by Toyama[7]. Toyama conjectured the modularity of completeness, but it was refuted by Klop and Barendregt[7]. Hsiang presented a revised conjecture but it was also refuted by Toyama[7].

It was in 1987 that the first positive results were discovered by Rusinowitch[5]:

- *$R_0 \oplus R_1$ is terminating and non-collapsing iff both R_0 and R_1 are so.*
- *$R_0 \oplus R_1$ is terminating and non-duplicant iff both R_0 and R_1 are so.*

However, these results are too restrictive to be applied in practice, because most of the practical systems are collapsing and/or duplicant.

In 1989, Toyama, Klop, and Barendregt[8] proved a reasonably practical result:

- *$R_0 \oplus R_1$ is complete and left-linear iff both R_0 and R_1 are so.*

This result is reasonably practical because, when we write a functional program as a term rewriting system, almost all systems can be written within the restriction of the completeness and the left-linearity.

However, term rewriting systems have important applications in automated theorem proving for first-order logic with equality, in which it is very common for a system to be non-left-linear. In this paper, we present a new result to solve this problem:

- **$R_0 \oplus R_1$ is simply-terminating iff both R_0 and R_1 are so**

where a system is *simply-terminating* if, intuitively, it's termination is proved with the simplification ordering method of Dershowitz[2].

2 Main Result

Consider a *partial ordering* (a transitive and irreflexive relation) $\succ$ on the set of terms $\mathcal{T}$. A partial ordering $\succ$ on $\mathcal{T}$ is *monotonic* if it possesses the *replacement* property,

$$s \succ t \quad \text{implies} \quad f(\ldots,s,\ldots) \succ f(\ldots,t,\ldots)$$

for all terms in $\mathcal{T}$. A monotonic partial ordering $\succ$ is a *simplification ordering* for $\mathcal{T}$ if it possesses the *subterm* property,

$$f(\ldots,t,\ldots) \succ t,$$

and the *deletion* property,

$$f(\ldots,t,\ldots) \succ f(\ldots,\ldots),$$

for all terms in $\mathcal{T}$. A simplification ordering $\succ$ on $\mathcal{T}$ *supports* a rewriting $s \rightarrow_R t$, if $s \succ t$. It supports a system R if it supports every rewriting $s \rightarrow_R t$ defined by R on $\mathcal{T}$. A term rewriting system R on $\mathcal{T}$ is *simply-terminating* if there exists a simplification ordering $\succ$ which supports R.

A term rewriting system R is *terminating* if there is no infinite rewrite sequence $t_0 \rightarrow_R t_1 \rightarrow_R \cdots$.

Theorem 2.1 (Dershowitz) *A simply-terminating system is, in fact, terminating.*

Theorem 2.2 *$R_0 \oplus R_1$ is simply-terminating iff both R_0 and R_1 are so.*

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† This paper is an extended abstract of [4].

Example Consider the following systems:

$$R_0 = \left\{ \begin{array}{l} x \cdot x \to x, \\ x \cdot (y + z) \to (x \cdot y) + (x \cdot z) \end{array} \right\}$$

$$R_1 = \{(x^{-1})^{-1} \to x\}$$

R_0 is shown to be simply-terminating by the recursive path ordering (which is a simplification ordering introduced by Dershowitz [1]). R_1 is also simply-terminating. Therefore, by our theorem, $R_0 \oplus R_1$ is simply-terminating. Note that the termination of $R_0 \oplus R_1$ cannot be proved by the three results by Rusinowitch, Toyama, et al. because R_0 contains collapsing, duplicant, and non-left-linear rules.

Many orderings suitable for semi-mechanical termination proof fall into the class of the simplification ordering. For example, the recursive path ordering (RPO) of Dershowitz, the path of subterm ordering (PSO) of Plaisted, and the recursive decomposition ordering (RDO) of Jouannaud are typical simplification orderings widely used [2,6]. It is known that there is a term rewriting system, say R_0, whose termination is proved with PSO but not with RDO nor RPO, and also there is a system, say R_1, whose termination is proved with RDO but not with PSO nor RPO [6]. Thus the termination of $R_0 \oplus R_1$ is proved with neither PSO nor RDO nor RPO. However, by our theorem, $R_0 \oplus R_1$ is simply-terminating, because both R_0 and R_1 are so.

3 Conclusion

We have presented a novel result on the modularity of the termination of term rewriting systems. The authors claim that not only the result itself is novel but also the class of the result is novel in that it focuses on the termination proof method, rather than explicitly restricted syntactic properties. Also, the result is independent of the confluence.

Proof with simplification ordering is one of the most powerful methods that are suitable for semi-mechanical termination proof. Therefore, our result is practically useful for the semi-mechanical termination proofs of 'modular' computer programs written as the set of term rewriting systems.

We feel that the result by Toyama, et al. and our result in this paper are practically the final solutions to the problem on the termination of the direct sum of term rewriting systems, and will close the series of the discussions so far. Our interest in future works will be in *non*-direct sums: under what conditions and strategies may the function symbols be shared among systems without losing termination and confluence? The problem is worth challenging, and some restricted results are already obtained [2]. Also, a novel approach to the modularity is proposed [3]. We believe that the solution would greatly contribute to the development of large-scale programs written as term rewriting systems, and thus enhance the mechanization of software engineering in the future.

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