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S_4-FACTORIZATION ALGORITHMS OF COMPLETE BIPARTITE GRAPHS

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Abstract. In this paper, a necessary condition for the existence of an S_4-factorization of $K_{m,n}$ is given. Several types of construction algorithms of S_4-factorization of $K_{m,n}$ are also given.

1. Introduction

Let S_4 be a *star* on 4 vertices and $K_{m,n}$ be a *complete bipartite graph* with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an S_4-*factor* if each component of F is isomorphic to S_4. If $K_{m,n}$ is expressed as an edge-disjoint sum of S_4-factors, then this sum is called an S_4-*factorization* of $K_{m,n}$.

2. S_4-factor of $K_{m,n}$

Theorem 1. $K_{m,n}$ has an S_4-factor if and only if (i) $m+n \equiv 0 \pmod 4$, (ii) $3n-m \equiv 0 \pmod 8$, (iii) $3m-n \equiv 0 \pmod 8$, (iv) $m \leqq 3n$ and (v) $n \leqq 3m$.

Corollary 1. $K_{n,n}$ has an S_4-factor if and only if $n \equiv 0 \pmod 4$.

3. S_4-factorization of $K_{m,n}$

Theorem 2. If $K_{m,n}$ has an S_4-factorization, then $K_{sm,sn}$ has an S_4-factorization for every positive integer s.

Notation 1. r,t,b : number of S_4-factors, number of S_4-components of each S_4-factor, and total number of S_4-components, respectively, in an S_4-factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_4-components of each S_4-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_4-components.

Trivial necessary conditions (T-conditions). $b=mn/3$, $t=(m+n)/4$, $r=4mn/3(m+n)$, $t_1=(3n-m)/8$, $t_2=(3m-n)/8$, $r_1=(3n-m)n/6(m+n)$, $r_2=(3m-n)m/6(m+n)$, $m \leqq 3n$ and $n \leqq 3m$.

Sufficient conditions. We consider the following three cases.

Case (1) $m=3n$: In this case, from Theorem 2, $K_{3n,n}$ has an S_4-factorization since $K_{3,1}$ is just S_4.

Case (2) $n=3m$: Obviously, $K_{m,3m}$ has an S_4-factorization.

Case (3) $m<3n$ and $n<3m$: In this case, let $x=(3n-m)/8$ and $y=(3m-n)/8$. Then from T-conditions, x and y are integers such that $0<x<m$ and $0<y<n$. We have $x+3y=m$ and $3x+y=n$. Hence it holds that $b=(x^2+3xy+y^2)+xy/3$, $t=x+y$, $r=(x+y)+4xy/3(x+y)$, $t_1=x$, $t_2=y$, $r_1=x-2xy/3(x+y)$ and $r_2=y-2xy/3(x+y)$. Let $z=2xy/3(x+y)$, which is a positive integer. And let $(x,3y)=d$, $x=dp$, $3y=dq$, where $(p,q)=1$. Then $dq/3$ is an integer and $z=2dpq/3(3p+q)$. Using these p,q,d, the parameters m and n satisfying T-conditions are expressed as follows:

Lemma 1. $(p,q)=1$ and $2dpq/3(3p+q)$ is an integer

===> (I) $m=3(p+q)(3p+q)s$, $n=(9p+q)(3p+q)s$ ($3p+q$:odd)
or $m=3(p+q)(3p+q)s'/2$, $n=(9p+q)(3p+q)s'/2$ ($3p+q$:even)
when $q/3$ is not an integer,

(II) $m=3(p+3q')(p+q')s$, $n=3(3p+q')(p+q')s$ ($p+q'$:odd)
or $m=3(p+3q')(p+q')s'/2$, $n=3(3p+q')(p+q')s'/2$ ($p+q'$:even)
when $q=3q'$ and $q'/3$ is not an integer,

(III) $m=(p+9q'')(p+3q'')s$, $n=3(p+q'')(p+3q'')s$ ($p+3q''$:odd)
or $m=(p+9q'')(p+3q'')s'/2$, $n=3(p+q'')(p+3q'')s'/2$ ($p+3q''$:even)
when $q=9q''$,

where s and s' are positive integers.

Notation 2. Let A and B be two sequences of the same size such as

A: $a_1, a_2, \ldots, a_u$

B: $b_1, b_2, \ldots, b_u$.

If $b_i = a_i + c$ $(i=1,2,\ldots,u)$, then we write B=A+c. If $b_i = ((a_i+c) \bmod w)$ $(i=1,2,\ldots,u)$, then we write B=A+c mod w, where the residuals a_i+c mod w are integers in the set $\{1,2,\ldots,w\}$.

Lemma 2. $(p,q)=1$ and $q/3$ is not an integer

$m=3(p+q)(3p+q)s$, $n=(9p+q)(3p+q)s$, where s is a positive integer

$\Longrightarrow$ $K_{m,n}$ has an S_4-factorization.

Proof. When s=1, the proof is by construction (Algorithm I). Let $x=(3n-m)/8$, $y=(3m-n)/8$, $t=(m+n)/4$, $r=4mn/3(m+n)$. Then we have $x=3p(3p+q)$, $y=q(3p+q)$, $t=(3p+q)^2$, $r=(p+q)(9p+q)$. Let $r_m=p+q$, $r_n=9p+q$, $m_0=m/r_m=3(3p+q)$, $n_0=n/r_n=3p+q$. Consider two sequences R and C of the same size $9(3p+q)$.

R: $1,1,1,2,2,2,\ldots,3(3p+q),3(3p+q),3(3p+q)$

C: $1,2,\ldots,9(3p+q)-1,9(3p+q)$.

Construct p sequences R_i such that $R_i=R+3(i-1)(3p+q)$ $(i=1,2,\ldots,p)$.

Construct p sequences C_i such that $C_i=(C+3(i-1) \bmod 9(3p+q))+9(i-1)(3p+q)$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $3(3p+q)$.

R': $r_1, r_2, \ldots, r_{3(3p+q)}$, where $r_i=(i-1)p+1 \bmod 3(3p+q)$ $(i=1,2,\ldots,3(3p+q))$

C': $c_1, c_2, \ldots, c_{3(3p+q)}$, where $c_i=n-((i-1)q \bmod q(3p+q))$ $(i=1,2,\ldots,3(3p+q))$.

Construct q sequences R_i' such that $R_i'=R'+3(i-1)(3p+q)+3p(3p+q)$ $(i=1,2,\ldots,q)$. Construct q sequences C_i' such that $C_i'=C'-(i-1)$ $(i=1,2,\ldots,q)$. Consider two sequences I and J of the same size.

I: $R_1, R_2, \ldots, R_p, R_1', R_2', \ldots, R_q'$

J: $C_1, C_2, \ldots, C_p, C_1', C_2', \ldots, C_q'$.

Then the size of I or J is 3t. Let i_k and j_k be the k-th element of I and J, respectively $(k=1,2,\ldots,3t)$. Join two vertices i_k in V_1 and j_k in V_2 with an edge (i_k,j_k) $(k=1,2,\ldots,3t)$. Construct a graph F with two vertex sets $\{i_k\}$ and $\{j_k\}$ and an edge set $\{(i_k,j_k)\}$. Then F is an S_4-factor of $K_{m,n}$.

Construct r_m sequences I_i such that $I_i=I+(i-1)m_0 \bmod m$ $(i=1,2,\ldots,r_m)$.

Construct r_n sequences J_j such that $J_j=J+(j-1)n_0 \bmod n$ $(j=1,2,\ldots,r_n)$.

Construct $r_m r_n$ S_4-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_m; j=1,2,\ldots,r_n)$. Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_4-factorization of $K_{m,n}$. By Theorem 2, $K_{m,n}$ has an S4-factorization for every positive integer s. ■

Lemma 3. $(p,q)=1$ and $q=3q'$ ($q'/3$ is not an integer)

$m=3(p+3q')(p+q')s$, $n=3(3p+q')(p+q')s$, where s is a positive integer

$\Longrightarrow$ $K_{m,n}$ has an S_4-factorization.

Lemma 4. $(p,q)=1$ and $q=9q''$

$m=(p+9q'')(p+3q'')s$, $n=3(p+q'')(p+3q'')s$, where s is a positive integer

$\Longrightarrow$ $K_{m,n}$ has an S_4-factorization.

REFERENCES

1. H. Enomoto, T. Miyamoto and K. Ushio, *C_k-factorization of complete bipartite graphs*, Graphs and Combinatorics, 4 (1988), pp. 111-113.

2. K. Ushio, *P_3-factorization of complete bipartite graphs*, Discrete Math., 72 (1988), pp. 361-366.

3. K. Ushio and R. Tsuruno, *P_3-factorization of complete multipartite graphs*, Graphs and Combinatorics, 5 (1989), pp. 385-387.

4. K. Ushio and R. Tsuruno, *Cyclic S_k-factorization of complete bipartite graphs*, to appear in "Proc. Second Inter. Conf. Graph Theory, 1989, San Francisco".