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Generalized Predicate Completion

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1. Introduction

Circumscription, proposed by J. McCarthy[3,4], is a formalism of non-monotonic reasoning. As the circumscription in first-order logic is generally a second-order sentence, its computation is difficult. Completion is an approach, proposed by K. Clark[1], to closed world reasoning which assumes that the given sufficient conditions on a predicate are also necessary. R. Reiter[6] has shown that for first-order theories in clausal form which are Horn in a predicate P, the circumscription of P logically subsumes the completion of P.

In this paper, we present a generalized completion of a predicate P, which is appropriate for the theories which are not Horn in P. The main results of this paper are:

(1) for non-overlapping first-order theories in clausal form (which may not necessarily be Horn in the predicate P), the circumscription of P logically subsumes the generalized completion of P;

(2) for non-overlapping first-order theories in clausal form which are collapsible wrt the predicate P, the generalized completion of P is even logically equivalent to the circumscription of P.

2. Circumscription

Circumscription [2,3,4,5,6] is an approach to the problem of non-monotonic reasoning, which augments formulas with a refinement of minimal inference. In this paper, we are particularly interested in clausal theories. A *clausal theory* is a set of clauses. A *clause* is a universally quantified disjunction of literals, written as $l_1 \vee \dots \vee l_n$, which is logically identified with $\forall x. (l_1 \vee \dots \vee l_n)$, whose variables are in $x = \{x_1, x_2, \dots, x_n\}$, in some cases a clause will be written as $\forall x. (Q_1 \supset Q_2)$, where Q_1 is a conjunction of literals and Q_2 a disjunction of literals.

Definition 1 Let T be a clausal theory, p and z distinct predicate symbols. The *circumscription* of p in T with parameter z, denoted by Circum(T; p; z), is defined as a second-order formula:

$$T \wedge \forall p', z'. [T(p', z') \wedge \forall x. (p'(x) \supset p(x)) \supset \forall x. (p(x) \supset p'(x))]$$

where p' and z' are predicate variables similar to p and z, $T(p', z')$ is the result of T substituting p' and z' for each occurrence of p and z, and x is the tuple of variables. If no z will be involved, Circum(T; p) is used for Circum(T; p; z). □

3. Predicate Completion

We shall proceed to the completion proposed by Clark [1]. Let T be a clausal theory. The *completion* of p in T, denoted by Comp(T; p), is the theory of T along with the completed definition of p and equality axioms (E1)~(E8) [2].

Comp(T; p) $T \wedge$
$$\forall x_1, \dots, x_n. [\, p(x_1, \dots, x_n) \supset E_1 \vee E_2 \vee \dots \vee E_k \,]$$

A clause is said to be *Horn* in a predicate symbol p (or a predicate P) *iff* it contains at most one positive literal on p. A clausal theory T is said to be Horn in p *iff* every clause of T is Horn in p. For clausal theories the predicate completion can be constructed heuristically. Besides this, as pointed by Reiter [6], the completion is sometimes implied by the circumscription.

Theorem 1 [6] *Let T be a clausal theory Horn in a predicate symbol p. Then* Circum(T; p) $\models$ Comp(T; p). □

4. Generalized Predicate Completion

As discussed previously, we know that for clausal theories which are Horn in p, the completion of p is implied by the circumscription of p. By some investigation, we understand this is not always the case for the clausal theories of general form. Then we shall try to augment the predicate completion with some refinement, called generalized predicate completion.

Let T be a theory consisting of a single clause (4.1).

$$\neg p(a) \supset p(b) \tag{4.1}$$

The completion of p in T, Comp(T; p), is then a theory of T along with the completed definition (4.2) of p and equality axioms (E1)~(E8) [2].

$$\forall x. [\, x = b \wedge \neg p(a) \equiv p(x) \,] \tag{4.2}$$

M_1 is a model of T minimal *wrt* $\leqq_p$, in which only $p(a)$ is evaluated true. It can be loosely described as $M_1 = \{p(a)\}$.

Since $M_1 \models p(a)$, then the *only-if-half* of the expression (4.2) is not true in M_1, hence $M_1 \not\models$ Comp(T; p). Therefore Circum(T; p) $\not\models$ Comp(T; p). □

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Proposition 1 *The completion is not always implied by the circumscription for any clausal theory.* □

It is plausible that a clause about a predicate symbol p contains more than one positive literal P. In [1], Clark pays attention upon only one positive literal P. Here we shall focus on each positive literal P respectively. Therefore we write a clause about p in the form of (4.3), which puts all positive literal P's explicitly on the right hand side of $\supset$.

Suppose that

$$l_1 \wedge \ldots \wedge l_m \supset p(t_{11}, \ldots, t_{1n}) \vee \ldots \vee p(t_{h1}, \ldots, t_{hn}) \quad (4.3)$$

is a clause, where l_i is any literal which is not negative one on p for any i, $1 \leqq i \leqq m$. The tuple $(t_{j1}, \ldots, t_{jn})$ is simply denoted by t_j, for any j, $1 \leqq j \leqq h$, the conjunction of $l_1 \wedge \ldots \wedge l_m$ by $\lambda(l)$. Then the clause (4.3) can be equivalently transformed into each of the following clauses about a predicate symbol p.

$$\lambda(l) \wedge \neg p(t_2) \wedge \ldots \wedge \neg p(t_h) \supset p(t_1)$$

$$\cdots\cdots$$

$$\lambda(l) \wedge \neg p(t_1) \wedge \ldots \wedge \neg p(t_{i-1}) \wedge \neg p(t_{i+1}) \wedge \ldots \wedge \neg p(t_h) \supset p(t_i)$$

$$\cdots\cdots$$

$$\lambda(l) \wedge \neg p(t_1) \wedge \ldots \wedge \neg p(t_{h-1}) \supset p(t_h)$$

Let $=$ be the equality relation, and $x_1, \ldots, x_n$ be variables not appearing in (4.3), simply denoted by x. If $y_1, \ldots, y_r$ are all variables in (4.3), simply denoted by y, those clauses can be equivalently transformed into the follows.

$$\exists y. [\, x = t_1 \wedge \lambda(l) \wedge \neg p(t_2) \wedge \ldots \wedge \neg p(t_h) \,] \supset p(x)$$

$$\cdots\cdots$$

$$\exists y. [x = t_i \wedge \lambda(l) \wedge \neg p(t_1) \wedge \ldots \wedge \neg p(t_{i-1}) \wedge \neg p(t_{i+1}) \wedge \ldots \wedge \neg p(t_h)] \supset p(x)$$

$$\cdots\cdots$$

$$\exists y. [\, x = t_h \wedge \lambda(l) \wedge \neg p(t_1) \wedge \ldots \wedge \neg p(t_{h-1}) \,] \supset p(x) \quad (4.4)$$

We call (4.4) the general forms of the clause (4.3).

Let T be a clausal theory. Suppose there are exactly k clauses in T, $k > 0$, about the predicate symbol p. Let

$$E_{11} \supset p(x)$$

$$\cdots\cdots$$

$$E_{kh_k} \supset p(x) \quad (4.5)$$

be $h_1 + h_2 + \cdots + h_k$ general forms of these k clauses. Each of E_{ij} will be an existentially quantified conjunction of literals as in (4.4),

$$\exists y. [x = t_i \wedge \lambda(l) \wedge \neg p(t_1) \wedge \ldots \wedge \neg p(t_{i-1}) \wedge \neg p(t_{i+1}) \wedge \ldots \wedge \neg p(t_h)].$$

The *generalized completed definition* of p, implicitly given by all of those k clauses, is the expression (4.6).

$$\forall x. [\, E_{11} \vee \cdots \vee E_{1h_1} \vee \cdots \vee E_{k1} \vee \cdots \vee E_{kh_k} \equiv p(x) \,] \quad (4.6)$$

When there is no clause about p, *i.e.*, $k = 0$, the generalized completed definition of p is:

$$\forall x. [\, \text{false} \equiv p(x) \,].$$

The *generalized* completion of p in T is a theory of T along with the generalized completed definition (4.6) of p and equality axioms (E1)~(E8)[1]. In order to distinguish from $\mathrm{Comp}(T; p)$ as T is not Horn in p, we use $\mathrm{Comp}_G(T; p)$ to denote the generalized completion of p in T.

$$\mathrm{Comp}_G(T; p) \quad T \wedge \forall x. [\, p(x) \supset E_{11} \vee \cdots \vee E_{1h_1} \vee \cdots \vee E_{k1} \vee \cdots \vee E_{kh_k} \,]$$

Definition 2 Let T be a clausal theory and p a predicate symbol. T is *non-overlapping wrt p iff* for each non-Horn clause C in p *of* T, P is not unifiable with P', where P and P' are any distinct positive literals on p in C. □

Theorem 2 *Let T be a clausal theory and p a predicate symbol. If T is non-overlapping wrt p then* $\mathrm{Circum}(T; p) \models \mathrm{Comp}_G(T; p)$. □

Corollary 2.1 *Let T be a clausal theory and p a predicate symbol. If* $P \supset Q$ *is not derivable from T for some clause* $Q \supset P$ *about p in T, then* $\mathrm{Th}(T) \subsetneq \mathrm{Th}(\mathrm{Comp}_G(T; p)) \subseteq \mathrm{Th}(\mathrm{Circum}(T; p))$, *where Q is a disjunction of literals and P an atom on p.* □

Corollary 2.2 *Let T be a clausal theory, p and z distinct predicate symbols. If T is non-overlapping wrt p then* $\mathrm{Circum}(T; p; z) \models \mathrm{Comp}_G(T; p)$. □

It is clear the converse of *Theorem* 2 is not always true.

Definition 3 Let T be a clausal theory and p a predicate symbol. T is *collapsible wrt p* if it consists of

(1) clauses containing no positive occurrences of p; and

(2) clauses containing no negative occurrences of p. □

Theorem 3 *Let T be a clausal theory and p a predicate symbol. If T is non-overlapping and collapsible wrt p, then* $\mathrm{Th}(\mathrm{Comp}_G(T; p)) = \mathrm{Th}(\mathrm{Circum}(T; p))$.

Corollary 3.1 *Let T be a clausal theory and p a predicate symbol. If T is Horn in p and collapsible wrt p, then* $\mathrm{Th}(\mathrm{Circum}(T; p)) = \mathrm{Th}(\mathrm{Comp}_G(T; p))\ (= \mathrm{Th}(\mathrm{Comp}(T; p)))$. □

5. Conclusion

In this paper, we have presented a generalized completion of a predicate P, which is appropriate for the theories in clausal form which are not Horn in P.

References

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