Syntactical Nonmonotonicity of Entailment Reasoning

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1. Introduction

How to reason based on incomplete knowledge is an important problem in knowledge engineering. In general, such reasoning should have a property known as nonmonotonicity: when we find a new fact and add it to our known knowledge, we may have to retract some consequences which are reasoned based on old knowledge.

Resent researches on nonmonotonic reasoning are based on the following three major approaches to extend the first order predicate calculus: introducing a minimization notion, introducing a modal operator, and introducing some inference rules [2].

In this paper, we show that reasoning based on entailment logic [3] is syntactically nonmonotonic. This nonmonotonicity is based on a fact that entailment logic is naturally nonmonotonic and therefore we need not introduce new logical primitives such as a minimization notion, a modal operator, or inference rules.

2. Propositional Entailment Logic

The entailment logic is constructed as a more natural logic system for our ordinary logical thinking than the classical mathematical logic [3]. A major property of the entailment logic is that it is "paradox-free", i.e., no implicational paradox is its logical theorem [3]. We call reasoning based on the entailment logic *entailment reasoning* [1].

A formal axiom system for the propositional entailment logic *Cm* is summarized as follows, where X, Y, and Z are syntactical variables.

Alphabet : (1) a denumerable set of proposition variable symbols; (2) the primitive logical connective symbols $\neg$(negation), $\wedge$(conjunction), and $\Rightarrow$(entailment); (3) brackets (and).

Formulas : (1) every proposition variable, called atomic formula, is a formula; (2) if X and Y are formulas then $\neg X$, $(X\wedge Y)$, $(X\Rightarrow Y)$ are formulas, where the outermost brackets of a formula can be omitted; (3) only those defined by (1) and (2) are formulas.

Logical Axiom Schemata :

A_1 $X\Rightarrow\neg\neg X$
A_2 $(X\Rightarrow(Y\Rightarrow Z))\Rightarrow(Y\Rightarrow(X\Rightarrow Z))$
A_3 $(Y\Rightarrow Z)\Rightarrow((X\Rightarrow Y)\Rightarrow(X\Rightarrow Z))$
A_4 $(\neg X\Rightarrow Y)\Rightarrow(\neg Y\Rightarrow X)$
A_5 $(X\wedge\neg Y)\Rightarrow\neg(X\Rightarrow Y)$
A_6 $X\Rightarrow(X\wedge X)$
A_7 $(X\wedge Y)\Rightarrow X$
A_8 $(X\wedge Y)\Rightarrow(Y\wedge X)$
A_9 $((X\Rightarrow Y)\wedge(X\Rightarrow Z))\Rightarrow(X\Rightarrow(Y\wedge Z))$
A_{10} $((X\wedge\neg(\neg Y\wedge\neg Z))\Rightarrow\neg(\neg(X\wedge Y)\wedge\neg(X\wedge Z)))$

Inference Rules :

R_1 From X and $X\Rightarrow Y$ to infer Y
R_2 From X and Y to infer $X\wedge Y$

Logical Theorems : (1) every axiom is a logical theorem; (2) the result of applying any of the inference rules to two logical theorems is a logical theorem; (3) only those defined by (1) and (2) are logical theorems.

Informal meanings of the logical connectives of entailment logic are as follows. $\neg X$ means "X does not hold". $X\wedge Y$ means "both X and Y hold". $X\Rightarrow Y$ is a logical abstraction of the sufficient conditional relation in our ordinary logical thinking and means "the fact that there is no such case as X holds and Y does not hold can be determined with neither determining whether X holds nor determining whether Y holds".

3. Nonmonotonicity of Entailment Reasoning

Below, we use P to denote a set of formulas $\{Z_1, Z_2, \cdots, Z_n\}$ and use q to denote a single formula.

Definition 1 A *deduction of q from P* based on a logic system is a finite sequence of formulas. Each member of the sequence is either a logical theorem, a member of P, or that obtained by applying any of the inference rules to some earlier members of the sequence. Each member of P is used at least once in an application of an inference rule. The last member of the sequence is q. All members of P are called the *premises* of the deduction and q is called the *consequence* of the deduction. We say P *syntactically entails* q or q *is provable from* P, write $P\vdash q$, if and only if there exists a deduction of q from P. We also say a logical theorem q is the consequence of a

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deduction from the empty set of premises and write $\vdash q$. □

Definition 2 For a given logic system, reasoning based on the logic is said to be *syntactically monotonic* if for any P and q, if $P \vdash q$ then $P \cup \{X\} \vdash q$ where X is any formula. □

The classical mathematical logic is naturally monotonic. Its monotonicity property is based on two fundamental facts. The first fact can be equivalently (because of the soundness and completeness of the logic) represented either in the notion of proof theory as "$P \cup \{X\} \vdash q$ iff $P \vdash X \rightarrow q$" (the deduction theorem) or in the notion of model theory as "$P \cup \{X\} \models_f q$ iff $P \models_f X \rightarrow q$" where $\rightarrow$ is the material implication and $\{X\} \models_f q$ denotes that q follows from X. Indeed, this fact shows both the syntactical relation $\vdash$ and the semantical relation $\models_f$ are equivalent to the material implication. The second fact can also be equivalently represented either in the notion of proof theory as "if $P \vdash q$ then $P \vdash X \rightarrow q$" or in the notion of model theory as "if $P \models_f q$ then $P \models_f X \rightarrow q$" where X is any formula. Thus, we can say that the monotonicity property of the classical mathematical logic is a direct result of modeling the sufficient conditional relation in our ordinary logical thinking using the material implication.

On the other hand, in the framework of entailment logic, the sufficient conditional relation in our ordinary logical thinking is modeled using the entailment notion. As a result, the deduction theorem for *Cm* does not hold but a stronger version of it holds.

Lemma 1 [3] $(X \Rightarrow Y) \Rightarrow (X \wedge Z \Rightarrow Y)$ is a logical theorem of *Cm*. □

Theorem 1 (The quasi-deduction theorem) In *Cm*, (1) if $P \vdash X \Rightarrow q$ then $P \cup \{X\} \vdash q$; (2) if $P \cup \{X\} \vdash q$ then $P \vdash (A \wedge Z \wedge X) \Rightarrow q$ where $A = A_1 \wedge A_2 \wedge \cdots \wedge A_m$ is the conjunction of all logical axioms occurring in the deduction of q from $P \cup \{X\}$ and $Z = Z_1 \wedge Z_2 \wedge \cdots \wedge Z_n$.

Proof (sketch) (1) Suppose $P \vdash X \Rightarrow q$. Then there exists a deduction of $X \Rightarrow q$ from P such as $d_1, d_2, \cdots, d_\ell = X \Rightarrow q$. Therefore, by inference rule R_1, $d_1, d_2, \cdots, d_\ell, X, q$ is a deduction of q from $P \cup \{X\}$. (2) Suppose $P \cup \{X\} \vdash q$. Then there exists a deduction of q from $P \cup \{X\}$ such as $d_1, d_2, \cdots, d_\ell = q$. We use induction over the length ℓ of the deduction. Let $\ell = 1$. Then $q \in \{A_1, A_2, \cdots, A_m\} \cup P \cup \{X\}$. If $q \in \{A_1, A_2, \cdots, A_m\}$, then by Lemma 1 and R_1, $q, \cdots, A \Rightarrow q, \cdots, (A \Rightarrow q) \Rightarrow (A \wedge Z \wedge X \Rightarrow q)$, $A \wedge Z \wedge X \Rightarrow q$ is a deduction of $A \wedge Z \wedge X \Rightarrow q$ from P. If $q \in P$, then $q, \cdots, Z \Rightarrow q, \cdots, (Z \Rightarrow q) \Rightarrow (A \wedge Z \wedge X \Rightarrow q)$, $A \wedge Z \wedge X \Rightarrow q$ is a deduction of $A \wedge Z \wedge X \Rightarrow q$ from P. If $q = X$, then $q, \cdots, X \Rightarrow q, \cdots, (X \Rightarrow q) \Rightarrow (A \wedge Z \wedge X \Rightarrow q)$, $A \wedge Z \wedge X \Rightarrow q$ is a deduction of $A \wedge Z \wedge X \Rightarrow q$ from P. Suppose now $\ell > 1$. By induction, we can prove $P \vdash (A \wedge Z \wedge X) \Rightarrow q$. □

Due to $X \Rightarrow (Y \Rightarrow X)$ is not a logical theorem of *Cm*, "if $P \vdash q$ then $P \vdash X \Rightarrow q$" does not hold for any X in *Cm*, and "if $P \vdash q$ then $P \vdash (A \wedge Z \wedge X) \Rightarrow q$" does not hold for any X in *Cm* too.

Consequently, in the framework of entailment logic, even if $P \vdash q$ holds, $P \cup \{X\} \vdash q$ does not necessarily hold. Therefore, entailment reasoning is syntactically nonmonotonic.

4. Concluding Remarks

We have shown in the notion of proof theory that entailment reasoning is syntactically nonmonotonic. This nonmonotonicity is based on the fact that both of the two fundamental facts which bring about the monotonicity property of the classical mathematical logic do not hold in the entailment logic and therefore the entailment logic is naturally nonmonotonic. As a result, in the framework of entailment logic, nonmonotonic reasoning can be achieved without introducing new logical primitives such as a minimization notion, a modal operator, or inference rules.

To investigate the nonmonotonicity property of entailment reasoning in the notion of model theory should based on a formal model of the entailment logic. However, it is not a completely solved problem to provide an adequate formal model for the full propositional entailment logic *Cm*. We have defined a subclass of *Cm* and given its algebraic model [1]. We are now investigating the semantical nonmonotonicity property of entailment reasoning based on this model.

Finally, we mention that relevance logic has a nonmonotonicity property which is similar to that of the entailment logic [4,5].

References

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