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Can Completion Entail Circumscription (Sometimes)?

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1. Introduction

Circumscription, a form of non-monotonic reasoning, proposed by *J. McCarthy* [3], of first-order predicate symbols are second-order formula. The formulas derivable from a theory T augmented by the circumscription of a predicate symbol P are just those true in all models of T minimal in P. In order to achieve this kind of minimal derivation, it is very important to find out some suitable first-order instances of circumscription. As there is actually such instances, we say the theory T is circumscriptively reducible [1] *wrt* the predicate symbol P. Completion of a theory, proposed by *Keith L. Clark* [4], is one way to represent closed world assumption [6] by assuming given sufficient condition about a predicate is also necessary. Shown in [7], circumscription of a predicate symbol P in a theory T Horn in P implies the completion of T *wrt* P. Here we will show that the completion of T *wrt* P entails the circumscription of P in T in the model-theoretical viewpoint.

2. Circumscription

Let T(P) be a theory with occurrences of P in the first-order language L. The *circumscription* of P in T(P) is defined by the following expression (1).

$\forall p.\ [T(p) \wedge [\forall x.\ p(x) \supset P(x)] \supset \forall x.\ P(x) \supset p(x)]$ (1)

Here P is an *n*-ary predicate symbol and *p* an *n*-ary predicate variable. T(*p*) is the conjunction of formulas of T with each occurrence of P replaced by *p*. Reasoning about the theory T under the closed world assumption has much in common with inferring from the theory T together with this formula. This enlarged theory, Circum(T; P), is called the circumscription of P in T.

3. Reducibility

Definition1

An *interpretation* I of a first-order language L consists of:

(1) a non-empty Herbrand universe D;

(2) $I[K]: D^n \rightarrow D$ if K is an *n*-ary function symbol;

$I[K]: D^n \rightarrow \{True, False\}$
if K is an *n*-ary predicate symbol.

$I[K^+] = \{a \in D^n \mid I[K](a) = True\}$ and $I[K^+] \subset D^n$;

$I[K^-] = \{a \in D^n \mid I[K](a) = False\}$ and $I[K^-] \subset D^n$.

We say an interpretation I [2] is a *model* of a formula G if G is evaluated to True in I and I is a model of theory T if every formula in T is satisfied by I. $T \models \beta$ is used to stand for that β is *entailed* by T, *i.e.*, β is true in every model of T.

A sub-interpretation N of M for T *wrt* a predicate symbol P, $N \leqq_P M$, is defined as:

1. N and M have a same domain;
2. For each symbol K (constant or function or predicate symbol), $M[K] = N[K]$, when $K \neq P$;
3. $N[P^+] \subseteq M[P^+]$.

A model M_0 of T is said to be minimal *wrt* P *iff* for any model M of T, if $M \leqq_P M_0$ then $M = M_0$. $T \models_P \beta$ is used to stand for that β is *minimally entailed* by T, *i.e.*, β is true in every model of T minimal *wrt* P.

Definition2

Let T be a first-order theory with occurrences of a predicate symbol P. Circum(T; P) is said to be *reducible iff* there is a first-order theory, written as $T_{Circum}(T; P)$, model-theoretically equivalent to Circum(T; P). That is, for any *wff* β in the first-order language,

$Circum(T; P) \models \beta$ *iff* $T_{Circum}(T; P) \models \beta$.

T is said to be *circumscriptively reducible* on P when Circum(T; P) is reducible.

4. Completion

Suppose that

$L_1 \wedge \ldots \wedge L_m \supset P(t_1, \ldots, t_n)$ (2)

is a clause about a predicate symbol P. Let = be the equality relation, and $x_1, \ldots, x_n$ be variables not appearing in the clause. (2) is equivalent to the clause

$x_1 = t_1 \wedge \ldots \wedge x_n = t_n \wedge L_1 \wedge \ldots \wedge L_m \supset P(x_1, \ldots, x_n)$

Finally, if $y_1, \ldots, y_r$ are the variables in (2), it is itself equivalent to

$(\exists y_1, \ldots, y_r)[x_1 = t_1 \wedge \ldots \wedge x_n = t_n \wedge L_1 \wedge \ldots \wedge L_m] \supset P(x_1, \ldots, x_n)$ (3)

we call this the *general form* of the clause.

Suppose there are exactly *k* clauses, $k > 0$, about the predicate symbol P. Let

$E_1 \supset P(x_1, \ldots, x_n)$
......
$E_k \supset P(x_1, \ldots, x_n)$ (4)

be *k* general forms of these clauses. Each of E_i will be an existentially quantified conjunction of literals as in (3). The *definition of P*, implicitly given by all of those *k* clauses, is

$(\forall x_1, \ldots, x_n)[E_1 \vee E_2 \vee \ldots \vee E_k \equiv P(x_1, \ldots, x_n)]$ (5)

The *if-half* of this definition is just the *k* general form

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clauses (4) grouped as a single implication. The *only-if-half* is the *completion axiom* for P.

5. Circumscription Implies Completion (Sometimes)

A first-order theory of clausal form T is said to be Horn in P *iff* every clause in T contains at most one positive literal on the predicate symbol P. Notice that the definition allows any number of positive literals in the clauses in T as long as their predicate symbols are distinct from P. Any such theory T may be partitioned into two disjoint sets:

T_P: those clauses in T containing exactly one positive literal in P, and

$T-T_P$: those clauses of T containing no positive (but possibly negative) literals on P.

The *completion* of T *wrt* P, Comp(T; P), is a theory of T with T_P replaced by the definition of P.

Theorem1[7]

Let T be a first order theory of clausal form, Horn in a predicate symbol P. Then

Circum(T; P)$\vdash$Comp(T; P)

i.e., the completion of T *wrt* P is derivable from the circumscription of P in T.

6. Completion Minimally Entails Circumscription (Sometimes)

As shown in [3], every *wff* circumscriptively inferred from Circum(T; P) is true in all models of T minimal *wrt* P. That is, Circum(T; P) is minimally entailed by T *wrt* P.

Lemma1 $T\models_p$Circum(T; P).

Lifschitz V. has presented that Circum(T; P) can be satisfied only by models of T minimal *wrt* P [5].

Lemma2 For any model M of T, if it satisfies Circum(T; P), then it is minimal *wrt* P.

Lemma3 Let T be a clausal theory Horn in P. Then $T\models_p$Comp(T; P).

【PROOF】It is sufficient to prove the *only-if-half* of (5) is satisfied by any model M_0 of T minimal in P. Firstly suppose the *only-if-half* of (5) is not satisfied by M_0. Then there is at least one $P(a_1,...,a_n)$ satisfied by M_0 but $(E_1\vee...\vee E_k)\theta$ not, and $\theta=\{x_1/a_1,...,x_n/a_n\}$. Thus a sub-interpretation M_0' of M_0 can be constructed in such a way that it agrees with M_0 on every symbol except for P, and $M_0'[P^+]=M_0[P^+]-(a_1,...,a_n)$. Obviously $M_0'<_P M_0$, and M_0' is a model of T. This contradicts with the minimality of M_0. Hence, the *only-if-half* of (5) is satisfied by any model M_0 of T minimal *wrt* P. ■

Together with above lemmas, we know that the models of Circum(T; P) are all of those minimal *wrt* P and Comp(T; P) is true in all of models of T minimal *wrt* P. Then Comp(T; P) is true in all of models of Circum(T; P). By the definition of $\models$, we have the following theorem.

Theorem2

Let T be a first-order theory of clausal form Horn in P. Then Circum(T; P)$\models$Comp(T; P).

Observing (5), Comp(T; P) is actually satisfied only by models of T minimal *wrt* P if T is Horn in P.

Lemma4 Let T be a clausal first-order theory Horn in P. Then any model M of Comp(T; P) minimal in P is a model of T minimal in P.

【PROOF】Let M be a model of Comp(T; P) minimal in P. Then M is a model of T. Now we shall prove that M is minimal in P. Suppose M_0 is a model of T with $M_0\leqq_P M$. Then for any $P(a_1,...,a_n)$ satisfied by M but not by M_0, $(E_1\vee...\vee E_k)\theta\supset P(a_1,...,a_n)$ has to be satisfied by M_0, where $\theta=\{x_1/a_1,..., x_n/a_n\}$. Then we can construct a proper sub-interpretation M' of M by:

$M'[K]=M[K]$ if $K\neq P$ and

$M'[P]=M_0[P]-(a_1,...,a_n)$.

M' is obviously a model of Comp(T; P) with $M'<_P M$. This contradicts the minimality of M. Therefore the model M of Comp(T; P) minimal in P is also a minimal model of T in P. ■

Theorem3

Let T be a clausal first-order theory Horn in P. Then Comp(T; P)$\models_p$Circum(T; P).

Summarize above two theorems, Comp(T, P) is entailed by Circum(T; P) and Circum(T; P) is minimally entailed by Comp(T; P). Because it is not always that a model of Comp(T; P) is a model of T minimal in P, then Comp(T; P) seems not always to be a suitable instance of Circum(T; P) even if T is a clausal theory Horn in P.

Theorem4

Let T be a clausal first-order theory Horn in P. Comp(T; P) is not a circumscriptively reduced theory of T *wrt* P, *i.e.*, T_{Circum}(T; P), if there is at least one model of Comp(T; P) is not a model of T minimal in P.

7. Conclusions

According to theorem4, a class of first-order theories, being of clausal form and Horn in a predicate symbol P, has been found. The circumscription of P in the theory of this class is minimally entailed by the completion of T *wrt* P. And it cannot be expected that for any theory of this class, the circumscription of a predicate symbol P is generally identified with its completion *wrt* P.

References

[1] B. H. Hou, A. Togashi and S. Noguchi *Circumscription Schemaのオーダーの縮小について* 情報処理学会第36回(昭和63年前期)全国大会P1339－1340

[2] Chang, C. L. and Lee, R. C. T. [1973] *Symbolic Logic and Mechanical Theorem Proving*, Academic Press, New York, 1973

[3] John McCarthy *Circumscription — A Form of Non-Monotonic Reasoning* AI 13(1980) 27–39

[4] Keith L. Clark *Negation as Failure* In Logic and Databases, H. Gallaire and J. Minker, Eds. Plenum, New York, pp 293–322

[5] Lifschitz, V., *Computing Circumscription* Proceedings Ninth International Joint Conference on AI, Los Angeles, CA (1985) 121-127

[6] Reiter, R. [1978] *On Closed World Data Bases* In Logic and Databases, H. Gallaire and J. Minker, Eds. Plenum, New York, pp 56–76

[7] Reiter, R. [1982] *Circumscription Implies Predicate Completion (Sometimes)* Proc. AAAI-82, 1982, pp 418–420