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On Space Complexity of Self-Stabilizing Leader Election in Population Protocol Based on Three-interaction

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Abstract: A population protocol is a distributed computing model for passively mobile systems, in which a computation is executed by interactions between two agents. This paper is concerned with an extended model, population protocol by interactions of three agents. Leader election is a fundamental problem in distributed systems, to select a central coordinator. Cai, Izumi, Wada(2011) showed that the space complexity of a self-stabilizing leader election for n agents is exactly n. This paper shows that the space complexity of the self-stabilizing leader election in a population protocol by interactions of three agents (SS-LE PP_3 for short) is exactly $\lceil \frac{n+1}{2} \rceil$.

1. Introduction

A population protocol is a distributed computing model formed by mobile agents with limited resource, which interact by a scheduler and change their own states [1]. Once an initial configuration is given, an execution of the system is determined by the order of interactions among agents. In this paper, we assume a scheduler is *adversarial* but *globally fair* (see Section 2).

The *Leader Election* (LE) problem is a problem of designating a single leader agent as the organizer of some task distributed among several agents. A PP for SS-LE, from an arbitrary initial configuration, eventually has to reach a configuration such that all of its successive configurations contain exactly one leader. According to the initial configuration, a PP for SS-LE has to decrease the number of leaders if it begins with more than one leader, while it has to appoint an agent to be a leader if it begins with no leader.

Cai et al. [3] showed that for a system of n agents, any PP for SS-LE requires at least n agent-states, and gave a PP with n agent-states for SS-LE. Mizoguchi et al. [4] gave an MPP for SS-LE with $\lceil (2/3)n \rceil + 1$ agent-states and two edge-states, and showed that any MPP for SS-LE with two edge-states requires more than $(1/2) \lg n$ agent-states.

In this paper, we are concerned with population protocol via three interaction model as an enhancement of the conventional model under complete graph of network [2]. We show that the space complexity is $\lceil \frac{n+1}{2} \rceil$.

Organization: This paper is organized as follows. We first give a definition of PP_3 model in section 2. Then we give a PP_3 for SS-LE with $\lceil \frac{n+1}{2} \rceil$ agent-states for the system of n agents in section 3. In section 4 we show that any PP_3 for SS-LE requires $\lceil \frac{n+1}{2} \rceil$ agent-states.

2. Models and Definitions

A population protocol by three interactions (PP_3) is defined by (Q, δ), where Q denotes a finite set of states and $\delta : Q \times Q \times Q \rightarrow Q \times Q \times Q$ is an update function of states by an interaction of a triple of agents.

A transition from a configuration C to the next configuration C' in an PP_3 is defined as follows. At the beginning, the scheduler chooses a triple of agents u_1, u_2, u_3. We assume that the scheduler can choose any triple. Suppose the states of the triple agents are p, q, r respectively and let $R : (p, q, r) \rightarrow (p', q', r')$ be a transition rule of δ. Then u_1, u_2, u_3 interact, writing as $C \xrightarrow{R} C'$, and the states of agents u_1, u_2, u_3 in C' are p', q', r' respectively, while all other agents keep their states in the transition. State of either node does not get necessarily changed which we call a silent transition. Transitions other than silent ones are called active.

An execution E is defined as an infinite sequence of configurations and transitions in alternation $C_0, R_0, C_1, R_1, \ldots$ such that for each i, $C_i \xrightarrow{R_i} C_{i+1}$. Like most of the literature on PP, we assume that the scheduler in an PP_3 is adversarial, but satisfying the strong global fairness, meaning that if a configuration C appears infinitely often in E then any possible transition from C must appear infinitely often in E as well. If $C \xrightarrow{R} C'$ for some R, we write $C \rightarrow C'$. The reflexive and transitive closure of $\rightarrow$ is denoted by $\xrightarrow{*}$. That is, $C \xrightarrow{*} C'$ means that a configuration C' is reachable (or can be generated) from a configuration C by a sequence of transitions of length more than of equal to 0. If any element in a set of states G can be generated from configuration C, we say that G can be generated from C, otherwise G cannot. G is said to be closed if for any element $p, q, r \in G$ and any transition $R \in \delta : (p, q, r) \rightarrow (p', q', r')$ indicates that $p', q', r' \in G$.

$\bot$ indicates an invalid state in which an agent is unable to join an interaction. The size of a configuration C (denoted as $|C|$) is size of agents with states other than $\bot$. For example, a a configu-

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ration (p, q, r, s) has size 4 and $(p, \perp, r, \perp)$ has size 2.

The *Leader Election* (LE) in PP_3 is the problem of assigning a special state of Q to exactly on agent, representing the "leader". A configuration $C \in Q^n$ is *legitimate* if C contains exactly one agent in the leader state, and so does any configuration C' satisfying $C \xrightarrow{*} C'$. Let $\mathcal{L}(\subseteq Q^n)$ denote the set of all legitimate configurations. A protocol for LE is *Self-Stabilizing* (SS) (with respect to $\mathcal{L}$) if the following condition holds:

For any configuration $C_0 \in Q^n$ and any execution $E = C_0 \xrightarrow{R_0} C_1 \xrightarrow{R_1} \ldots$ starting from C_0, there is an $i \geq 0$ such that $C_i \in \mathcal{L}$. We use the term PP_3 for SS-LE to indicate a self-stabilizing population protocol by three interactions for leader election.

3. Upper Bound of the Space Complexity

Theorem 3.1. *There exists a PP_3 using $\lceil \frac{n+1}{2} \rceil$ agent states which solves the SS-LE for n agents.*

To begin with, we give a Protocol 1 corresponding to the situation $n \equiv 1 \bmod 2$ and then we modify the Protocol 1 to suit for the situation $n \equiv 0 \bmod 2$.

$$m = \lceil \tfrac{n+1}{2} \rceil = \begin{cases} \frac{n+1}{2} & \text{if } (n \equiv 1 \bmod 2) \\ \frac{n+2}{2} & \text{if } (n \equiv 0 \bmod 2) \end{cases}$$

Protocol 1.

$Q = \{q_0, q_1, \ldots, q_{m-1}\}$, where q_0 denotes the leader state.

$\delta = \{$

$R_1 : (q_0, q_0, q) \rightarrow (q_0, q_{m-1}, q)$ for $q \in Q$,

$R_2 : (q_i, q_i, q_i) \rightarrow (q_i, q_i, q_{i-1})$, in cases $i \neq 0$

$R_3 : (q_i, q_j, q_k) \rightarrow (q_i, q_j, q_k)$, in cases other than R_1 and R_2

$\}$

We define a set of configurations $\mathcal{L} \subseteq Q^n$, such that $C \in \mathcal{L}$ if $\gamma_0(C) = 1$ and $\gamma_i(C) = 2$ *for* $i \in \{1, 2, \ldots, m-1\}$ where $\gamma_i(C)$ denotes the number agents in state q_i in C for each $i = 0, 1, \ldots, m-1$.

Lemma 3.2. *$\mathcal{L}$ is the set of legitimate configurations.*

Proof. According to Protocol 1, no agent is able to change its state after reaching $C \in \mathcal{L}$, there would always be a unique agent with leader state afterwards. □

Lemma 3.3. *For any configuration $D \in Q^n$, there exists an execution that satisfies $D \xrightarrow{*} C$ such that $C \in \mathcal{L}$.*

Proof. By the definition of Protocol 1, if $\gamma_i(E) \geq 2$ $(i \neq 0)$ holds for $E \in Q^n$, then $\gamma_i(F)$ holds for any $F \in Q^n$ such that $E \xrightarrow{*} F$. In case that $\gamma_0(D) = 0$, then we claim that q_0 is generated as follows: While $\gamma_0(D) = 0$, there exists q_i satisfying that $\gamma_i(D) \geq 3$, applying $R_2(D \xrightarrow{R_2} E)$ results $\gamma_i(E) = \gamma_i(D) - 1$ and $\gamma_{i-1}(E) = \gamma_{i-1}(D)+1$. Repeated by applying R_2, we get an agent with q_0 and the number of agents with q_0 would never reduce to 0 again according to the transition rules. Finally transform state q_0 to q_{m-1} by applying R_1 if more than one agent in state of q_0 exist. Then, replenish the agents of state size less than 2. See Figure 1. We obtain the claim. □

By combining Lemma 3.2 and Lemma 3.3, we obtain that Protocol 1 is able to solve SS-LE PP_3 problem with size $n \equiv 1 \; mod \; 2$.

For situation $n \equiv 0 \; mod \; 2$, we simply modify the Protocol 1 as following:

$\delta = \{$

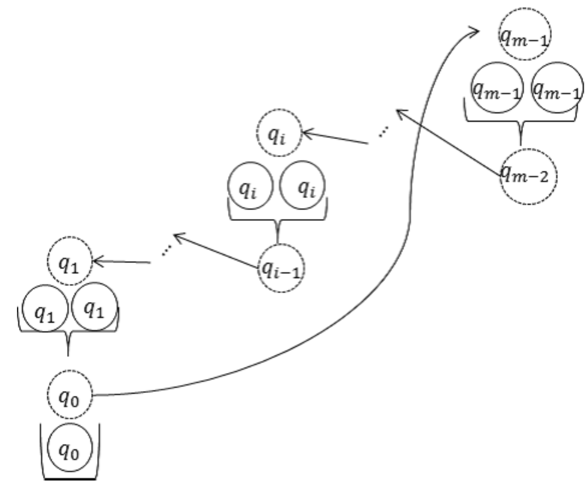

Fig. 1 Process to reach Legitimate Configuration

$R_1 : (q_i, q_i, q) \rightarrow (q_i, q_{(i-1) mod\; m}, q)$ for $q \in Q$, in cases $i = 0$ or $m-1$

$R_2 : (q_i, q_i, q_i) \rightarrow (q_i, q_i, q_{i-1})$, in cases $0 < i < m-1$

$R_3 : (q_i, q_j, q_k) \rightarrow (q_i, q_j, q_k)$, in cases other than R_1 and R_2

$\}$

Proofs are similar to the previous case which we would omit here.

4. Lower Bound of the Space Complexity

Theorem 4.1. *No SS-LE PP_3 for n agents exists with agent states size less than $\lceil \frac{n+1}{2} \rceil$.*

Lemma 4.2. *Let G be a finite subset of states, and suppose that C is a (sub)configuration of a SS-LE PP_3 which cannot generate G. Then, at least one of the following two conditions holds:*

(i) The complement of G (denoted by $\overline{G}$) is closed.

(ii) There exists a configuration $C' \subseteq C$ and a set $G' \supset G$ such that $|C| - 2 \leq |C'|$, $|G| + 1 \leq |G'|$ and configuration C' cannot generate G'.

Proof. We will show that if (i) does not hold then (ii) holds. Suppose that $\overline{G}$ is not closed, meaning that there exists a transition $(p, q, r) \rightarrow (p', q', r')$ such that $p, q, r \in \overline{G}$ and at least one of $p', q', r' \notin \overline{G}$. We consider the following two cases:

Case 1. One element of p, q, r cannot be generated from C. Without loss of generality, we may assume p cannot be generated from C. Then, we obtain the condition (ii) by setting $C' = C$ and $G' = G \cup \{p\}$.

Case 2. All elements of p, q, r can be generated from C. Then we consider the following three cases:

Case 2.1. All the three elements can be generated from C at the same time. It implies that G can be generated from C by $(p, q, r) \rightarrow (p', q', r')$. Contradiction.

Case 2.2. It is not the Case 2.1 and only two elements of p, q, r can be generated from C at the same time. Without loss of generality, we may assume p and q are generated. Now, we mask these two agents with $\perp$, and C' be the subconfiguration excluding these two agents, i.e., $|C'| = |C| - 2$. Then C' cannot generate setting $G' = G \cup \{r\}$, otherwise it is Case 2.1, we obtain condition (ii).

Case 2.3. Only one element can be generated at the same time. Without loss of generality, we may assume p is generated. Then we obtain the condition (ii) by setting $G' = G \cup \{q, r\}$ and $|C'| = |C| - 1$, with masking the agent with state p.

□

Lemma 4.3. *In a SS-LE protocol, the set of states excluding leader state would never be closed.*

Proof. If such kind of set exists, a configuration initialized by elements only in the set would be unable to generate the leader state which results in a contradiction. □

Proof of Theorem 4.1. Suppose problem in $n = 2n'$ ($2n' + 1$ respectively) agents can be solved by a protocol using state set whose size equals $n' = \lceil \frac{n-1}{2} \rceil$, we say Q: $\{q_0, q_1, ..., q_{n'-1}\}$ where q_0 denotes the leader state. Let C be a legitimate configuration, which only contains one leader state. We set C_0 by masking the agent with q_0 in C with $\perp$, thus $|C_0|=2n' - 1$ ($2n'$ respectively). Then set $G_0 = \{q_0\}$. By property of legitimate configuration, C_0 cannot generate G_0. Also by Lemma 4.3, we know $\overline{G}$ is not closed. By Lemma 4.2, we can obtain a configuration C_1 and a set G_1 satisfying $|C_1| \geq |C_0| - 2$, $|G_1| \geq |G'| + 1$ and configuration C_1 cannot generate G_1.

In a similar way, recursively by applying Lemma 4.2 $n' - 1$ times, we get that $C_{n'-1}(|C_{n'-1}| = 1$ or $2)$ cannot generate $G_{n'-1}(|G_{n'-1}| = n')$ that equals Q. Contradiction. □

5. Conclusions

This paper showed that the space complexity of SS-LE PP_3 is $\lceil \frac{n+1}{2} \rceil$. In a similar way, we can show that the space complexity of SS-LE PP_k is $\lceil \frac{n-1}{k-1} \rceil + 1$.

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