Electronic Preprint for Journal of Information Processing Vol.20 No.3

Regular Paper

Enumerating 3D-Sudoku Solutions over Cubic Prefractal Objects

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Received: September 1, 2011, Accepted: December 16, 2011

Abstract: We consider three-dimensional extensions of the Sudoku puzzle over prefractal objects. The prefractal objects we use are 2nd-level cubic approximations of two 3D fractals. Both objects are composed of 81 cubic pieces and they have 9×9-grid appearances in three orthogonal directions. On each object, our problem is to assign a digit to each of the 81 pieces so that it has a Sudoku solution pattern in each of the three 9×9-grid appearances. In this paper, we present an algorithm for enumerating such assignments and show the results.

Keywords: enumeration, 3D-Sudoku, fractal

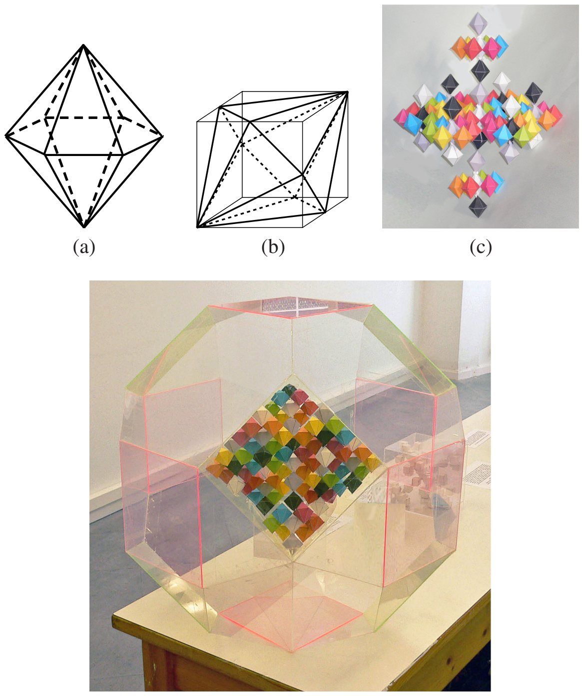
1. Introduction

In 2007, Tsuiki presented a "Fractal Sudoku Sculpture" in **Fig. 1** [3]. It is a three-dimensional prefractal object in that it has the form of the 2nd-level approximation of the hexagonal bipyramid fractal, and is composed of 81 hexagonal bipyramid pieces. Interestingly, it has the property that it has 9×9-grid appearances in *six* directions if a pair of opposite directions is counted once. This sculpture is colored in a "Sudoku coloring." That is, the 81 pieces are colored in 9 colors so that, in all of the six 9×9-grid appearances, each row, each column, and each of the nine 3×3 blocks contains all the 9 colors. He enumerated all the Sudoku colorings of this object and showed that there are 140 colorings modulo change of colors [2].

In this enumeration, at first he used a computer with a simple backtracking search algorithm, but finally he derived all of them by hand. His experiences show that the six-ways Sudoku constraint is so strong that there are only a small number of solutions. Therefore, a simple backtracking search algorithm works in a realistic time, and one can even enumerate them by hand.

On the other hand, the enumeration of Sudoku grids is done by Felgenhauer and Jarvis in 1995 and they obtained the number 18,383,222,420,692,992 modulo change of digits [1]. Here, there are so many solutions that it looks impossible to obtain this number without a devised algorithm. Their algorithm divides patterns of the top band (i.e., the top three rows) into equivalence classes so that patterns belonging to the same class have the same number of full-grid Sudoku-solution extensions.

In this paper, we consider a problem which exists between these two. We consider the 2nd-level approximation of the same fractal object which consists of 81 cubes instead of hexagonal bipyramids as **Fig. 3** (a) shows. It has square projections in three orthogonal directions, that is, it is an imaginary cube in Ref. [4]. Thus, it has 9×9-grid appearances in *three* directions. Our problem is to enumerate assignments of digits to the 81 pieces so that it has a Sudoku solution pattern in each of the three 9×9-grid appearances. Here, since we consider the three-ways Sudoku condition instead of the six-ways, we have more solutions than the



(d)

Fig. 1 (a) A hexagonal bipyramid whose faces are isosceles triangles with the height 3/2 of the base. (b) The object (a) put into a cube. As this figure shows, it has a square appearance from each of the 12 faces (i.e., in 6 directions). (c) "Fractal Sudoku Sculpture," which consists of 81 hexagonal bipyramid pieces. (d) The object (c) put in a transparent polyhedron with 12 square colored faces. As this picture shows, it has 9×9-grid appearances with Sudoku solution color patterns from each of the 12 square faces.

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six-ways case and enumeration with a simple search algorithm takes a long time with a PC. On the other hand, since we have additional constraints to the original Sudoku problem, the method of Felgenhauer and Jarvis cannot be applied directly.

We solved this problem by modifying the method of Felgenhauer and Jarvis for Sudoku solution grids. We found permutations on the top band which preserve the number of full-grid extensions for our 3D-Sudoku problem. As the result, we have 5,065,278 solutions modulo change of digits. There is another prefractal object which consists of 81 cubes (Fig. 3 (b)) and it is shown that it has the same number of 3D-Sudoku solutions. Finally, it is also shown that some imaginary cubes which are composed of 81 cubes but which do not have prefractal structures also have the same number of 3D-Sudoku solutions.

As we will see, one can express our three-ways Sudoku condition with a colored 9×9 Sudoku grid. Therefore, one can play it as a puzzle on a color-printed paper. That is, some of the cells are filled with digits from the beginning and the player fills the rest so that the three-ways Sudoku condition is satisfied. In addition, there are $5{,}065{,}278 \times 9!$ solution patterns, which is large enough to make different puzzles. Note that there are many 3D extensions of the Sudoku puzzle [5]. Usually, such extensions use the surface of a cube and thus extending the grid on which a puzzle is defined. In contrast, our 3D-Sudoku puzzle uses the same grid as the original Sudoku puzzle and it has additional constraints which are naturally expressed through a three-dimensional structure.

In the next section, we explain our 3D-Sudoku problems. In Section 3, we review the algorithm for enumerating Sudoku solution grids by Felgenhauer and Jarvis, and in Section 4, we give our algorithm for enumerating 3D-Sudoku solutions. We give the result of computation and study some non-prefractal cases in Section 5.

2. 3D-Sudoku Solutions over Cubic Prefractal Objects

Consider the operation on a cube to cut it into $n \times n \times n$ small cubes and select n^2 of them so that all of them can be seen from each of the three surface directions of the cube. We call an object which is composed of such a configuration of n^2 cubes a cubic imaginary cube of order n. Note that, if we fix a face of the cube to start with, a cubic imaginary cube of order n exists corresponding to a $n \times n$ Latin square where a number in a cell specifies the depth position of the corresponding cube. For the case $n = 3$, one can see that there are twelve 3×3-Latin squares and there are two order-3 cubic imaginary cubes (H) and (T) in **Fig. 2** if we identify rotationally equivalent ones. Now, for (H) and (T), we apply this operation to obtain nine cubes from one cube to each of the nine cubes. In this way, we have two objects which are cubic imaginary cubes of order 9. They are the objects on which we consider our problem. Figure 3 (a) shows the object obtained from (H) and Fig. 3 (b) shows the object obtained from (T).

If we repeat this procedure infinitely, we have fractal objects which we call the hexagonal bipyramid fractal for (H) and the triangular antiprismoid fractal for (T) [3]. Therefore, we call each object in Fig. 3 the 2nd level cubic prefractal approximation of the corresponding fractal. Note that the convex hull of the hexagonal

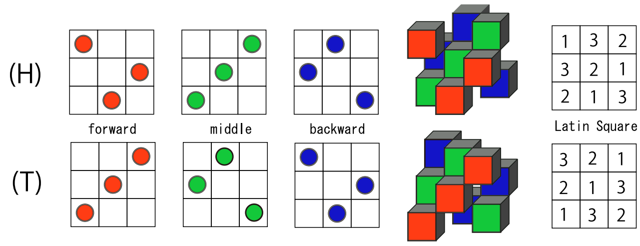
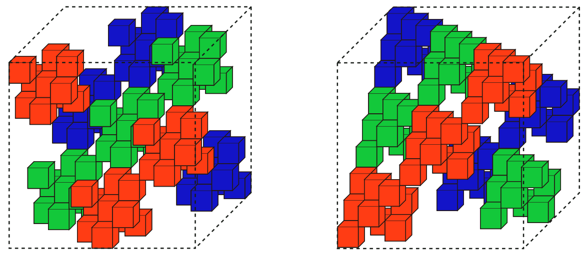


Fig. 2 The two cubic imaginary cubes of order 3.



(a) (b)

Fig. 3 (a) The 2nd-level cubic prefractal approximation of the hexagonal bipyramid fractal. (b) That of the triangular antiprismoid fractal.

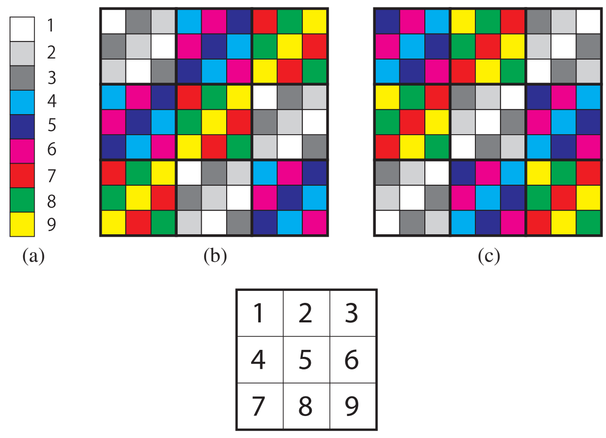


Fig. 4 (a) Coloring used in (b) and (c). (b) Depth of the nine cubes represented as colors for the object in Fig. 3 (a). (c) The same as (b) for the object in Fig. 3 (b). (d) Address of blocks and address inside a block.

bipyramid fractal is a hexagonal bipyramid in Fig. 1 (a) and if we replace the 81 cubes of Fig. 3 (a) with hexagonal bipyramids, we have the object in Fig. 1 (c).

On each of these two prefractal objects, we consider the problem of assigning 9 digits to the 81 cubes so that it has a Sudoku solution pattern in each of the three 9×9-grid appearances. In **Figs. 4** (b) and (c), one surface direction of the object is fixed and the depth of each cube is represented as colors. Through this coloring, one can see that these two problems are equivalent to the original Sudoku puzzle with the additional constraint that cells with the same depth (i.e., the same color in Figs. 4 (b) and (c)) have different digits. Note that other colorings of the 81 cells with 9 colors so that each color appears nine times defines an extension of the original Sudoku puzzle with the additional constraint that different digits are assigned to cells with the same color. Among them, if the coloring forms a 9×9 Latin square by identifying a color with a digit, then it corresponds to a cubic imaginary cube of order nine and one can consider this puzzle as a 3D-Sudoku puzzle on this order-9 cubic imaginary cube. Our two objects are special cases that objects have 3D prefractal structures.

We formalize this problem in the following way. Let $\Sigma = \Gamma = \{1,2,3,4,5,6,7,8,9\}$. Though Σ and Γ are the same set, we use Σ for addressing cubes and Γ for digits. We use the viewpoint of Figs. 4 (b) and (c) and give an address (b, c) to each cube. Here, b designates the block and c designates the position of a cube in the block according to the addressing in Fig. 4 (d). We call a map $\Sigma \times \Sigma \to \Gamma$ a numbering.

Definition 1 A numbering $\gamma : \Sigma \times \Sigma \to \Gamma$ is a (hexagonal bipyramid) 3D-Sudoku solution if

(a) $\gamma|_{(b,\Sigma)}$ is a surjection for each $b \in \Sigma$.

(b) Let $A_1 = \{1,2,3\}$, $A_2 = \{4,5,6\}$, and $A_3 = \{7,8,9\}$. For each pair $(i, j) \in \{1,2,3\} \times \{1,2,3\}$, $\gamma|_{A_i \times A_j}$ is a surjection.

(c) Let $B_1 = \{1,4,7\}$, $B_2 = \{2,5,8\}$, and $B_3 = \{3,6,9\}$. For each pair $(i, j) \in \{1,2,3\} \times \{1,2,3\}$, $\gamma|_{B_i \times B_j}$ is a surjection.

(d) Let $C_1 = \{1,6,8\}$, $C_2 = \{3,5,7\}$, and $C_3 = \{2,4,9\}$. For each pair $(i, j) \in \{1,2,3\} \times \{1,2,3\}$, $\gamma|_{C_i \times C_j}$ is a surjection.

Here, (a), (b), (c), and (d) correspond to the conditions on blocks, rows, columns, and depths, respectively. Therefore, numberings which satisfy (a), (b), and (c) are the original Sudoku solution grids.

In the same way, if one uses $C'_1 = \{3,5,7\}$, $C'_2 = \{2,4,9\}$, $C'_3 = \{1,6,8\}$ instead of C_1, C_2, and C_3, then we have the definition of a triangular antiprismoid 3D-Sudoku solution. However, one can see that $\{C_1, C_2, C_3\} = \{C'_1, C'_2, C'_3\}$. It means that Fig. 4 (c) is obtained from Fig. 4 (b) by changing the colors. Therefore, a numbering is a hexagonal bipyramid 3D-Sudoku solution if and only if it is a triangular antiprismoid 3D-Sudoku solution. Thus,

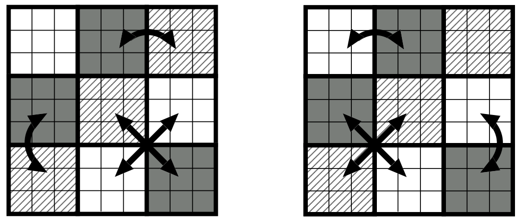
Proposition 2 The numbers of 3D-Sudoku solutions for the triangular antiprismoid case and the hexagonal bipyramid case are equal.

3. Enumeration of Sudoku Solution Grids

We call the top three rows of a 9×9-grid the top band. It consists of 27 cells. We briefly review the algorithm by Felgenhauer and Jarvis for enumerating Sudoku solution grids [1]. They defined an equivalence relation on the assignments of digits to the top band in such a way that any two elements in the same class have the same number of extensions to full-grid Sudoku solutions. They obtained 44 equivalence classes and then computed the number of extensions for a representative of each equivalence class. In this way, they obtained the number of Sudoku solution grids.

They generated such an equivalence relation by considering operations on top-band numberings which preserve Sudoku-solution extensions. There are two kinds of operations.

The first operations are those which have conditions on applicable numberings. For example, if a top-band numbering $\gamma_1 : \{1,2,3\} \times \Sigma \to \Gamma$ satisfies $\gamma_1(1,1) = \gamma_1(2,4) = a$ and $\gamma_1(1,4) = \gamma_1(2,1) = b$, then the top-band numbering $\gamma_2 : \{1,2,3\} \times \Sigma \to \Gamma$ obtained by swapping a and b in these four cells have the same number of extensions to the full-grid solution because any assignment to the other six rows which forms a Sudoku solution grid with γ_1 forms a Sudoku solution grid with γ_2, and vice versa. This does not hold for 3D-Sudoku solutions because the condition of the depth is not taken into account, and one can check with a computer that this operation on top-band numberings does not preserve the number of 3D-Sudoku solution extensions.



(a) (b)

Fig. 5 (a) Permutation β_1. (b) Permutation β_2.

The other ones are those operations defined as permutations of the cells. For example, the operation to swap Block 2 and Block 3 of a numbering of the top-band preserves the number of full-grid Sudoku-solution extensions because by also swapping Blocks 5 and 6 and swapping Blocks 8 and 9, one obtains a transformation between their full-grid Sudoku-solution extensions. For our 3D-Sudoku problem, this full-grid transformation does not preserve the depth-equivalence and is not a transformation between full-grid extensions. However, one can see that if we define a permutation β_1 over Σ^2 as in **Fig. 5** (a), that is, swapping each of the pairs of blocks: $(2,3)$, $(5,9)$, $(6,8)$, $(4,7)$, then a 3D-Sudoku solution is mapped to a 3D-Sudoku solution. Therefore, the top-band permutation to swap Block 2 and Block 3 preserves the number of 3D-Sudoku solution extensions.

4. Enumeration of 3D-Sudoku Solutions

We found this kind of permutations of the top-band $\{1,2,3\} \times \Sigma$ which can be extended to a whole-grid permutation which preserves 3D-Sudoku solutions. We give a formal definition as follows.

Definition 3 For a numbering $\delta : \{1,2,3\} \times \Sigma \to \Gamma$, a *3D-Sudoku extension* of δ is a 3D-Sudoku solution γ such that $\gamma|_{\{1,2,3\}\times\Sigma} = \delta$.

Definition 4 A permutation η over Σ^2 *preserves 3D-Sudoku solutions* if $\gamma \circ \eta$ is a 3D-Sudoku solution for every 3D-Sudoku solution γ.

Lemma 5 If a permutation η over Σ^2 preserves 3D-Sudoku solutions, then η^{-1} preserves 3D-Sudoku solutions.

Proof: If n is the order of η, then $\eta^{-1} = \eta^{n-1}$. ∎

Definition 6 A permutation α over $\{1,2,3\} \times \Sigma$ *preserves 3D-Sudoku extensions* if α is a restriction of a permutation over Σ^2 which preserves 3D-Sudoku solutions.

Proposition 7 Suppose that a permutation α over $\{1,2,3\} \times \Sigma$ preserves 3D-Sudoku extensions. Then, for every numbering $\delta : \{1,2,3\} \times \Sigma \to \Gamma$, δ and $\delta \circ \alpha$ have the same number of 3D-Sudoku extensions.

Proof: Since $\alpha : \{1,2,3\} \times \Sigma \to \{1,2,3\} \times \Sigma$ preserves 3D-Sudoku extensions, there is a permutation η over Σ^2 which preserves 3D-Sudoku solutions such that $\alpha = \eta|_{\{1,2,3\}\times\Sigma}$. Suppose that $\gamma : \Sigma^2 \to \Gamma$ is a 3D-Sudoku extension of $\delta : \{1,2,3\} \times \Sigma \to \Gamma$. Then, $\gamma \circ \eta$ is a 3D-Sudoku solution such that $(\gamma \circ \eta)|_{\{1,2,3\}\times\Sigma} = \gamma \circ \alpha = \delta \circ \alpha$. Therefore, $\gamma \circ \eta$ is a 3D-Sudoku extension of $\delta \circ \alpha$. On the other hand, if γ' is a 3D-Sudoku extension of $\delta \circ \alpha$, then

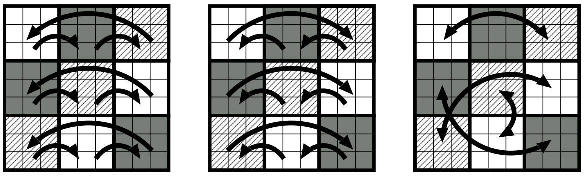


Fig. 6 Permutations generated by β_1 and β_2.

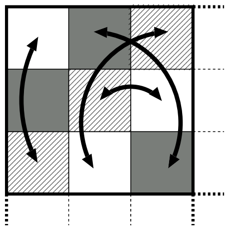


Fig. 7 Permutation ι_3.

$\gamma' \circ \eta^{-1}$ is a 3D-Sudoku extension of δ. Therefore, this correspondence is one to one. ∎

The set of permutations on $\{1, 2, 3\} \times \Sigma$ which preserve 3D-Sudoku extensions forms a group. We want to find its subgroup U which is large enough. It acts on numberings of $\{1, 2, 3\} \times \Sigma$, and divides them into equivalence classes such that all the members of a class have the same number of 3D-Sudoku extensions. Therefore, we find permutations on $\{1, 2, 3\} \times \Sigma$ which preserve 3D-Sudoku extensions. Note that, by Definition 6, such a permutation exists corresponding to a permutation of Σ^2 which preserves 3D-Sudoku extensions and maps $\{1, 2, 3\} \times \Sigma$ to itself.

Let $\phi_1 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,3\,2\,7\,9\,8\,4\,6\,5 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 2\,1\,3\,8\,7\,9\,5\,4\,6 \end{pmatrix}$ be permutations over Σ. As Fig. 5 shows, we define permutations β_i $(i = 1, 2)$ over Σ^2 as

$$\beta_i((b, c)) = (\phi_i(b), c) \quad (i = 1, 2).$$

The permutation β_i $(i = 1, 2)$ preserves 3D-Sudoku solutions because all the 3×3 blocks of this prefractal have the same structure and blocks with the same depth are mapped to blocks with the same depth as Fig. 5 shows. Note also that β_i maps the top band to itself $(i = 1, 2)$. As compositions of β_1 and β_2, one obtains permutations in **Fig. 6**. In this way, β_1 and β_2 generate a group of order six.

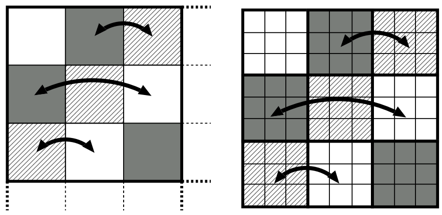
One can also define permutations ι_i $(i = 1, 2)$ over Σ^2 as

$$\iota_i((b, c)) = (b, \phi_i(c)) \quad (i = 1, 2).$$

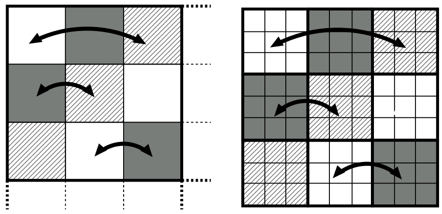
Note that ι_i $(i = 1, 2)$ preserves blocks and makes the same permutation in each block. Note that any permutation which preserves blocks maps the top-band to itself. Therefore, we can consider more permutations. Let $\phi_3 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 7\,9\,8\,4\,6\,5\,1\,3\,2 \end{pmatrix}$ be a permutation over Σ and define a permutation ι_3 over Σ^2 as follows. See **Fig. 7**.

$$\iota_3((b, c)) = (b, \phi_3(c)).$$

The permutations ι_1, ι_2, and ι_3 generate a group H of order 18. In each block, an element of this group yields a permutation of the three rows and a permutation of the three columns, and these two permutations determine an element of H. Therefore, this group is an index-2 subgroup of the product $S_3 \times S_3$ of two symmetric groups, and each element of H is an even permutation if it is considered as a member of S_6.



(a) τ_1



(b) τ_2

Fig. 8 Permutations τ_1 and τ_2 which swap row and depth.

We consider one more permutation on Σ^2. As Fig. 3 (a) shows, our object has mirror symmetry and the reflection to swap the depth and the row maps the object to itself. Thus, the permutation τ_1 to swap the depth and the row address of each cube preserves 3D-Sudoku solutions (**Fig. 8**). We define a permutation $\phi_4 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,3\,2\,6\,5\,4\,8\,7\,9 \end{pmatrix}$ over Σ and define τ_1 as

$$\tau_1((b, c)) = (\phi_4(b), \phi_4(c)).$$

We can similarly define a permutation τ_2 over Σ^2 based on the other prefractal object Fig. 3 (b). That is, we define $\phi_5 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 3\,2\,1\,5\,4\,6\,7\,9\,8 \end{pmatrix}$ and τ_2 as

$$\tau_2((b, c)) = (\phi_5(b), \phi_5(c)).$$

However, $\phi_5 = \phi_4 \circ \phi_2 \circ \phi_1$ and we do not need it as a generator of our group.

Let $\alpha_\beta = \beta|_{\{1,2,3\}\times\Sigma}$ for $\beta \in O = \{\beta_1, \beta_2, \iota_1, \iota_2, \iota_3, \tau_1\}$. Each $\beta \in O$ preserves 3D-Sudoku solutions and maps $\{1, 2, 3\} \times \Sigma$ to itself. Therefore, we have the following proposition.

Proposition 8 Each α_β for $\beta \in O$ preserves 3D-Sudoku extensions.

5. The Result of Computation

Let U be the group generated by α_β for all $\beta \in O$. It has order 216. We define equivalence $\sim$ on $\{1, 2, 3\} \times \Sigma \to \Gamma$ as $\delta \sim \delta'$ if $\delta' = \delta \circ \alpha$ for some $\alpha \in U$. If $\delta \sim \delta'$, then δ and δ' have the same number of 3D-Sudoku extensions. Therefore, we enumerated 3D-Sudoku solutions in two steps. (A) List all the equivalence classes of the numberings on $\{1, 2, 3\} \times \Sigma$ by $\sim$. We record the cardinality and a representative element of each class, (B) Compute the number of 3D-Sudoku extensions for a representative of each equivalence class.

In Ref. [1], it is shown that there are 2,612,736 numberings of the top-band which satisfy the original Sudoku condition restricted to the top-band modulo change of digits. Since our depth-condition restricted to the top-band is weaker than the block-condition, we started with these 2,612,736 numberings and divided them into equivalence classes by ∼. As the result, we obtained 12,542 equivalence classes as the result of Step (A). As the result of Step (B), we had 5,065,278 numberings.

Result 9 (1) There are $5{,}065{,}278 \times 9! = 1{,}838{,}088{,}080{,}640 \simeq 1.838 \times 10^{12}$ 3D-Sudoku solutions both for the triangular antiprismoid case and the hexagonal bipyramid case.
(2) There are 5,065,278 3D-Sudoku solutions modulo change of digits for both of the cases.

This result was firstly computed in 2008, with the speed that Step (A) took 15 seconds and Step (B) took 4 minutes with a PC (Intel Celeron 2.50 GHz, with 472 MB Memory). It is verified with a simple backtracking search algorithm running on a PC (Intel Core i7-M680 2.80 GHz, with 4 GB Memory) in 3 hours in 2011.

As is mentioned in Section 2, there are many cubic imaginary cubes of order nine, which are the objects on which one can consider 3D-Sudoku problems. Finally, we study some of them. Let L be the set of 3×3 Latin squares. The cardinality of L is 12. For $A, B \in L$, we define an order-9 cubic imaginary cube $\mathrm{S}(A, B)$ which is composed of nine blocks located according to A and each block is composed of nine cubic pieces located according to B. For example, $\mathrm{S}(\mathrm{H}, \mathrm{H})$ is the hexagonal bipyramid cubic prefractal and $\mathrm{S}(\mathrm{T}, \mathrm{T})$ is the triangular antiprismoid cubic prefractal for H and T the Latin squares in Fig. 2.

Lemma 10 There are ten order-9 cubic imaginary cubes of the form $\mathrm{S}(A, B)$ for $A, B \in L$ if we identify rotationally equivalent ones.

Proof: The set L is divided into two sets L_H and L_T, which are the sets of Latin squares generating order-3 cubic imaginary cubes (H) and (T), respectively. An element of L_H determines one of the four ways (H) is placed in a cube and is characterized by the line connecting the pair of opposite cube-vertices contained in (H). We denote by G_H the set of four lines connecting opposite vertices of a cube. Similarly, an element of L_T determines one of the eight ways (T) is placed in a cube, and is characterized by the cube-vertex surrounded by the three cube-vertices contained in (T). For any two pairs of lines (l_1, l_2) and (g_1, g_2) such that $l_1 \neq l_2 \in G_H$ and $g_1 \neq g_2 \in G_H$, there is a rotation σ of the cube such that $g_1 = \sigma(l_1)$ and $g_2 = \sigma(l_2)$. Thus, all the $\mathrm{S}(H_1, H_2)$ for $H_1 \neq H_2 \in L_H$ are rotationally equivalent. Therefore, there are only two objects of the form $\mathrm{S}(H_1, H_2)$ for $H_1, H_2 \in L_H$ modulo rotational equivalence, that is, $H_1 = H_2$ and $H_1 \neq H_2$. Similarly, there are two of the form $\mathrm{S}(T_1, H_1)$, two of the form $\mathrm{S}(H_1, T_1)$, and four of the form $\mathrm{S}(T_1, T_2)$ for $T_1, T_2 \in L_T$ and $H_1 \in L_H$. ∎

Proposition 11 For $A, B \in L$, the number of 3D-Sudoku solutions of $\mathrm{S}(A, B)$ is the same as the number in Result 9.

Proof: Let S be the set of operations on L to swap two rows or to swap two columns. One can see that any two elements of L are convertible through repeated applications of operations in S. On the other hand, for $f \in S$ and $A' \in L$, the object $\mathrm{S}(A', B)$ and the object $\mathrm{S}(f(A'), B)$ have the same number of 3D-sudoku solutions. Therefore, $\mathrm{S}(A, B)$ and $\mathrm{S}(B, B)$ have the same number of 3D-sudoku solutions. Since the object $\mathrm{S}(B, B)$ is rotationally equivalent to S(H, H) or S(T, T), this proposition holds. ∎

Note that there are also other order-9 cubic imaginary cubes with the same number of 3D-Sudoku solutions as the number in Result 9. For example, one can consider the object obtained from the hexagonal bipyramid prefractal by moving the depth 1 cubes to depth 2 locations and depth 2 cubes to depth 1 locations without changing their rows and columns.

Acknowledgments We would like to express our thanks to Tomonori Ito for his 3D-Sudoku software prototype with which one can enjoy our 3D-Sudoku puzzle under an excellent user interface.

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