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On the Influence of Approximation in a Lattice for Deterministic Volatility Models

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In this research we investigate both the influence of the approximation and the fitting to the market price in a Lattice Construction method over Deterministic Volatility Models (DVM) using statistical tests. Li (2000/2001) proposed the Lattice Construction method which can express the market price flexibly using appropriate DVMs. However, this method has the implicit influence of approximation caused by recombining which is visible for DVM Lattice and is not common for Black-Scholes Lattice model. As a novelty approach we propose a new verification methodology of determining best DVM regarding the influence of approximation and the fitting to the market price.

1. Introduction

The most essential and well known option valuation model is Black-Scholes model (BS model)[1]. This model adopts geometric Brownian motion and its volatility of equity return is constant. Applying the Black-Scholes' formula to actual option market prices is obtained the implied volatility curve. The attained implied volatility curve is not flat. That's why so many volatility models were suggested to provide the flexibility of the volatility.

To represent the volatility of the model more flexibly, Deterministic Volatility Models (Dupire (1994)[2], Derman and Kani (1994)[3], Rubinstein (1994)[4] and so on) have been proposed. Later on, different numerical approaches solving the DVMs are suggested. Li (2000/2001)[5] proposed an interesting solution for DVMs using Lattice Construction method. His method is based on a new algorithm for constructing implied binomial trees. In the DVMs the Local Volatility function plays main role. Our work examines four kinds of such functions (including BS model). In an important early contribution to our paper, Mawaribuchi, Miyazaki and Okamoto (2009)[6] have chosen the parameters of Local Volatility functions in a way that the difference between the Lattice model price and the market price is the smallest possible. In that way it can be verified which DVM is closer to the market values. In Mawaribuchi, Miyazaki and Okamoto (2009), however, it is not studied the influence of approximation caused by recombining which is visible for DVM Lattice and is not common for BS Lattice model. In our paper we examine this approximation together with a study about fitting to the market price in DVMs. The parameters in the different Local Volatility functions are estimated using optimization in the Lattice Construction method and they are the best ones regarding the Lattice method. At the same time they contain the influence of the approximation caused by recombining. In our paper, it is considered by way of Monte Carlo simulation technique which doesn't have influence of recombining to represent DVMs.

It is difficult to discuss about approximation in DVMs because they don't have closed-form solutions like BS model. In our work we propose new statistical tests of verifying the influence of approximation caused by recombining together with fitting to the market price. These statistical tests use the option model price distribution generated by Monte Carlo simulation technique (which doesn't include the approximation of recombining). The parameters of the Local Volatility functions in Monte Carlo method are estimated using the Lattice method. It means that these parameters are optimal ones for Lattice method with the influence of recombining but they are not the best ones for Monte Carlo method. In that way we effectively capture the influence of approximation caused by recombining in DVMs.

This paper is organized as follows: In section 2 is presented the Lattice Construction method over Deterministic Volatility Models. We explain the purpose of the analyses and the methods of verifying the results (we propose statistical tests). Section 3 is dedicated to empirical analyses and demonstrates the main results. The last section contains the conclusion.

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Table 1 Deterministic Volatility Models.

DVM	Local Volatility	
1P	$\sigma(S_t,t)=a$	BS Model
2P	$\sigma(S_t,t)=aS_t^b$	CEV Model
3P	$\sigma(S_t,t)=c+a\left\{1-\tanh\left[b\left(\frac{S_t-S_0}{S_0}\right)\right]\right\}$	Li Model
5P	$\sigma(S_t,t)=c+a\left\{1-\tanh\left[b\left(\frac{S_t-S_0}{S_0}\right)\right]\right\}$ $+d\left\{1-\mathrm{sech}\left[e\left(\frac{S_t-S_0}{S_0}\right)\right]\right\}$	MMO Model

2. Deterministic Volatility Models and their Lattices

2.1 Deterministic Volatility Models

The stock price process of Deterministic Volatility Model follows the stochastic differential equation (1).

$$\frac{dS_t}{S_t}=rdt+\sigma(S_t,t)\,d\hat{W} \tag{1}$$

where, S_t, r, $\sigma(\cdot)$ and $d\hat{W}$ are underlying asset, risk-free interest rate, local volatility (this function includes the underlying asset S_t and certain period of discretization t), and Brownian motion under risk-neutral measure, respectively. The DVM is specified by the functional form of the local volatility. Four kinds of such models are listed in **Table 1** (including 1 parameter model which is like BS model)(Refer to Mawaribuchi, Miyazaki and Okamoto (2009)).

2.2 Option pricing and Lattice Construction method in DVM (Li (2000/2001))

Having in mind that our paper is based on European call and put options, in equation (2) we show the valuation formulas in these cases.

$$\begin{array}{l}\text{Call Price}=\mathrm{e}^{\text{-rT}}\int_0^\infty \max(S_T-K,0)\,f(S_T)\,dS_T\\ \text{Put Price}=\mathrm{e}^{\text{-rT}}\int_0^\infty \max(K-S_T,0)\,f(S_T)\,dS_T\end{array} \tag{2}$$

where r, S_T and $f(S_T)$ are risk-free rate, equity price at the maturity, and probability density function at the maturity, respectively. In order to utilize equation (2) it is useful to derive numerically (using Lattice Construction method) the density function at the maturity - $f(S_T)$.

To construct the binomial lattice for each DVM from Table 1, it is convenient to adopt Li algorithm that proposes setting both up and down transition probabilities at 50%. Hoshika and Miyazaki (2008)[7] noticed that the robustness of the Li algorithm is higher compared to that of Derman and Kani (1994). In another paper Mawaribuchi, Miyazaki and Okamoto (2009) demonstrated that 5 parameter model can express flexibly various kinds of option market prices. The conceptual graphic of the lattice is shown in **Figure 1**. S_t^i denotes the underlying asset price at time t, which falls on the i-th node (counting up, starting from the top of the t period). On the figure is shown the skewness of the DVM Lattice.

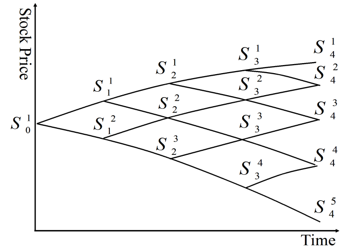

Fig. 1 Basic structure of Li Model Binomial Lattice.

Li algorithm

Equations (3) give the asset prices between two consecutive time periods (time interval is Δt). The stock prices in the current period (t) are expressed by using the stock prices from the previous period ($t-1$).

$$\begin{array}{l}S_t^1=S_{t-1}^1\left[1+r\Delta t+\sigma\left(S_{t-1}^1,t\right)\sqrt{\Delta t}\right],\\ S_t^{t+1}=S_{t-1}^t\left[1+r\Delta t-\sigma\left(S_{t-1}^t,t\right)\sqrt{\Delta t}\right],\\ S_t^{i+1}=\frac{1}{2}\left\{\begin{array}{l}S_{t-1}^i\left[1+r\Delta t-\sigma\left(S_{t-1}^i,t\right)\sqrt{\Delta t}\right]\\ +S_{t-1}^{i+1}\left[1+r\Delta t+\sigma\left(S_{t-1}^{i+1},t\right)\sqrt{\Delta t}\right]\end{array}\right\}.\quad(i\neq 0,t)\end{array} \tag{3}$$

The first and the second equations generate the top and the bottom stock paths in the lattice and the third equation describes all the stock paths inside the lattice (Figure 1). The third equation contains a value of $\frac{1}{2}$ which is the approximation of recombining.

Figure 2 shows geometrically the stock prices obtained by Li algorithm regarding BS Lattice and DVM Lattice. Looking at the stock price S one may

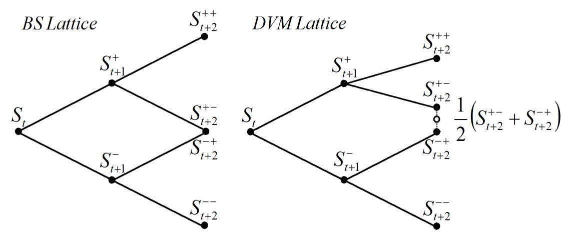

Fig. 2 BS Lattice and DVM Lattice.

notice that under suffix is the time, upper suffix are up-movements of the previous stock price (+) and down-movements of the previous stock price (-). For example, S_{t+2}^{+-} is stock price at time $t+2$ and the previous movements of it are up and down. In BS Lattice, up and down range is constant in all stock price periods because the volatility is constant. Stock prices S_{t+2}^{+-} and S_{t+2}^{-+} are the same values in BS Lattice but in DVM Lattice they are not equal because the volatility is changes. So, as from Equation (3), Li algorithm uses approximation of recombining of $\frac{1}{2}\left(S_{t+2}^{+-}+S_{t+2}^{-+}\right)$.

2.3 Purpose of the Analyses

Main point in this paper is to determine which DVM is the best regarding the influence of the approximation and the fitting to the market price. In **Table 2**, stock price process equation of Black-Scholes type is expressed in (A). The corresponding Local Volatility function is for 1 parameter model. In Figure 2, Black-Scholes Lattice doesn't contain the influence of approximation caused by recombining. The graphic for BS Lattice is symmetric. The influence is derived from discrete approximation (time interval is Δt). BS model has closed-form solution and the discrete approximation can be derived easily. In the same time BS model cannot express the market price flexibly and therefore, as demonstrated in Mawaribuchi, Miyazaki and Okamoto (2009) the fitting is not the best one. Stock price process equation of DVM type is expressed like in (B). The corresponding Local Volatility function is for 2, 3 and 5 parameter models. In Figure 2, DVM Lattice contains the influence of approximation caused by recombining. But DVM doesn't have closed-form solutions and it is difficult to estimate the influence of the approximation. For that reason we adopt the Monte Carlo method which doesn't include such characteristic to estimate the influence of the approximation. Another specific thing about DVMs (compared to BS model) is that the ones which have more parameters in their Local Volatility function have better fitting to the market price.

Table 2 The concept of this study.

		Continuous model	Lattice model	Closed-form solutions	Approximation
BS	(A)	$\frac{dS_t}{S_t}=rdt+\sigma dW$	1P model	○ (BS formula)	discretization
DVM	(B)	$\frac{dS_t}{S_t}=rdt+\sigma\left(S_t,t\right)dW$	2P model 3P model 5P model	× (unattainable)	discretization and recombining

Our work proposes new statistical tests of verifying the influence of the approximation and the fitting to the market price.

2.4 Method of the Analyses

2.4.1 Fitting to the market price in the Lattice Construction method

In option pricing theory, it is important to check whether the adopted model can express the market price flexibly. Below we demonstrate the optimization in the Lattice Construction method.

In aim to minimize the sum of square errors (the differences between the model prices and their corresponding market prices) it is convenient to calibrate the parameters of each model and identify which model well replicates the cross-sectional option market prices. The smaller the minimized sum of square errors the better the calibration. Totally 6 kinds of out-of-the-money (OTM) options are used in the calibration and these are OTM1 (the strike price is the closest to the current equity price), OTM2 (the strike price is the second closest to the current equity price), OTM3 (the strike price is the third closest to the current equity price) call and put options. In this study, the models are estimated in a way to minimize their objective functions (equation (4)). Once the model is identified with the minimum objective functional value, the estimated parameters are the best for the model to replicate the cross-sectional option market prices.

Objective function

$$Min\sum_{i=1}^{6}(P_i^{'}-P_i)^2/6 \tag{4}$$

where P and P' are option market price and option model price, respectively. i indicates type of option and $i = 1, 2, 3, 4, 5$ and 6 represent Call OTM1, Call OTM2, Call OTM3, Put OTM1, Put OTM2 and Put OTM3, in order.

2.4.2 Monte Carlo method and Statistical test 1, test 2 and test 3

We propose new statistical tests of determining best DVM regarding both the influence of approximation and the fitting to the market price.

As mentioned more up, we use Monte Carlo method in this paper for the proposed statistical tests. There are typically three steps: generating sample paths, evaluating the payoff along each path, and taking an average to obtain the option prices.

Formula (1) takes part in applying the Monte Carlo method to the problem. Expression (1) transforms to (5) using the Ito's lemma.

$$S_t = S_0 e^{\int_0^t \left(r - \frac{\sigma^2(S_u,u)}{2}\right) du + \int_0^t \sigma(S_u,u) dW_u} \tag{5}$$

where S_t, S_0, r, $\sigma(S_u, u)$ and W are the equity price at t, the initial stock price, the risk-free rate, the Local Volatility function with the parameters estimated by the Lattice method and the Brownian motion, respectively. Using this Local Volatility function we can carve out the influence of approximation. The discrete version of (5) is (6).

$$S_t = S_0 e^{\sum_{u=0}^{n-1} \left(r - \frac{\sigma^2(S_{u\Delta t}, u\Delta t)}{2}\right) \Delta t + \sum_{u=0}^{n-1} \sigma(S_{u\Delta t}, u\Delta t) z_u \sqrt{\Delta t}} \tag{6}$$

Here $z \in N(0, 1)$, and z is independent and identically distributed (i.i.d.) variable. Δt is the time interval. $n = \frac{t}{\Delta t}$ is the discretization period (n is integer).

In that way the local volatility in each of the periods of the discretization (Δt) in the discrete model are applied. And the time interval Δt of the Monte Carlo simulation is set to be the same one like in the Lattice method.

In this work we use 1000 Monte Carlo paths and the discretization periods are 30. For each of these we generate 5000 iterations and we take the average value from these 5000 generations (simulations). Having these 5000 values, later in the paper we present the Monte Carlo model price distribution for all of the estimated DVMs.

Below, the ideas behind the conducted tests are presented. Test 3 is the main object of our study. We determine the best DVM taking into account both the influence of approximation and fitting to the market price. Test 1 and test 2 are used to estimate test 3.

Test 1

Test 1 is a comparison between the Lattice model price and the Monte Carlo model price. We make a verification concerning the influence of approximation caused by recombining for DVM. If the Lattice model price is rejected by the Monte Carlo model price distribution, it means that the influence of approximation caused by recombining for DVM is not negligible.

Test 2

In test 2 we conduct a study about the fitting between the market price and the Monte Carlo model price. If the market price is within the 99% confidence interval then the price is not rejected. Otherwise, the market price is rejected.

Test 3

Test 3 is a result from test 1 and test 2. If test 1 and test 2 are not rejected then test 3 is not rejected. That means market price, Lattice model price and Monte Carlo model price are very close each other. Then the fitting between the Lattice model price and market price is very good and also the influence of approximation caused by recombining for DVM is negligible.

3. Empirical Analyses

3.1 Data and setting of the analyses

For estimating the different DVMs we use 6 kinds of options - 3 Put options and 3 Call options. The options are monthly contracts and their maturities are 15 business days. The result of the analysis is attained one for each month. The covered data period is from June 2003 until July 2007, or totally 50 months. The number of discretization periods in the Lattice and Monte Carlo methods is 30.

3.2 Fitting to the market price of several DVMs in Lattice Construction method

It is important to check which DVM is closest to the market price. Therefore we suggest a study about best fitting DVM to the market price. In **Figure 3**, we consider the average absolute difference between the Lattice model price and

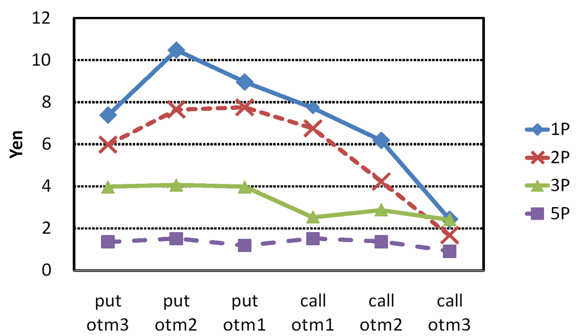

Fig. 3 Absolute difference of Lattice and Market price (average).

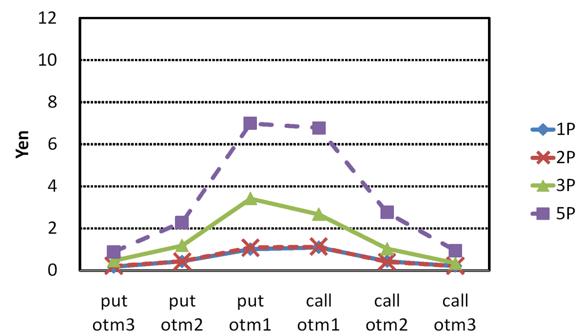

Fig. 4 Absolute diffrence of Monte Carlo and Lattice price (average).

the market price of 50 months data. We pick up the best parameters minimizing the objective function and graphically demonstrate the results.

Figure 3 shows the results about the four observed models. 1 parameter model (which is like BS model) is a simple one and it can't capture the market price flexibly. 2 parameter model includes skewness to a certain level and it behaves better than 1 parameter model. 3 parameter model represents skewness more flexibly than 2 parameter model and its fitting is better. From all of the observed DVMs it is obvious that the difference between 5 parameter model and market price is the lowest. This result justifies the existence of skewness and kurtosis in actual option market, and the fitting of 5 parameter model is the best among the observed four DVMs.

3.3 The influence of approximation caused by recombining in the Lattice Construction method

In **Figure 4**, we consider the average absolute difference between the Lattice model price and the Monte Carlo model price of 50 months data (The value of each month is the average of 5000 generations (simulations)). From the figure, we can see that the absolute difference is not negligible for the DVMs that have more parameters in their Local Volatility functions. The reason is that the influence of the approximation of recombining prevails with the increase in the flexibility of the model. In particular, the influence of approximation of OTM1 options is relatively large.

3.4 An example of the tests 1, 2 and 3

To illustrate the idea of the proposed test 1, test 2 and test 3 in the case of 5 parameter model, we show one interesting example (out of 50 months data) in **Figures 5**. The contract of observation is January 2004, put otm2 option.

In Figure 5, we plot the Monte Carlo model price distribution (for 5000 simulations) and its average, Lattice model price and the market price. The horizontal axis contains the option prices and the vertical axis gives the respective frequencies. The market price is close to the average of Monte Carlo model prices and within the 99% confidence interval. These models can express the market price flexibly. Test 2 is not rejected. Taking into account that test 1 is not rejected then test 3 is also not rejected. From this observation, the average value from the Monte Carlo model price, the Lattice model price and the market price are very close each other. In this case, 5 parameter model is the best DVM regarding both the influence of the approximation and the fitting to the market price.

3.5 Results and implications of the tests 1, 2 and 3

Table 3 shows the results of tests 1, 2 and 3 from section 2.4.2. We consider 50 months data and a 99% confidence interval. Digit 0 means that the model is not rejected and the different from 0 values show in how many month (out of 50) the model is rejected. The table can be divided into 3 parts. Test 1 is equivalent to Figure 4 in section 3.3 and verifies the influence of recombining which is visible for DVM Lattice. Test 2 is similar to Figure 3 in section 3.2 and

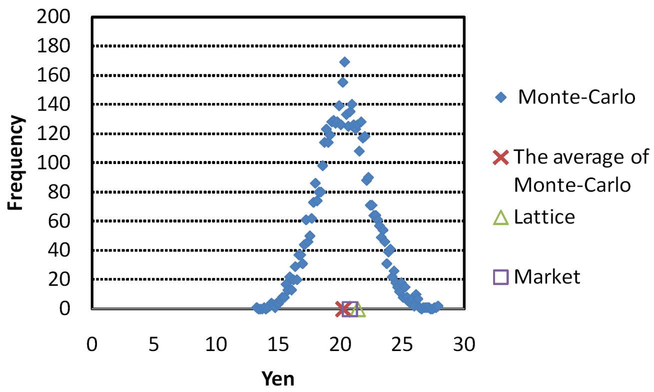

Fig. 5 5P, 2004/01, put otm2.

it concerns the difference between the market price and the model price (Test 2 compares the market price and the Monte Carlo model price, Figure 3 compares market price and Lattice model price). Test 3 is combination of test 1 and test 2. It demonstrates which DVM is the best regarding both the influence of the approximation and the fitting to the market price.

We focus on the third part of the Table 3 that concerns the influence of approximation and fitting to the market price. If test 3 is not rejected it means that the market price, the Lattice model price and the Monte Carlo model price are very close. Looking at all of the 6 options totally it was observed that the rejected data decreases gradually from 1 parameter to 5 parameter models. Thinking about the fitting and recombining together, we demonstrate the advantage of extending the model (in this case to 5 parameter model). In particular, in OTM2 and OTM3 it is obvious that there is an improvement caused by the extension.

4. Conclusions

In this paper, we propose new statistical tests using the Monte Carlo simulation technique which reveal the best DVM regarding both the influence of approximation caused by recombining and the fitting to the market price.

When we consider both the influence of approximation caused by recombining and the fitting to the market price, 5 parameter model demonstrates the best results. In particular, in OTM2 and OTM3, we recognize the visible improvement caused by the extension of the model.

Table 3 Two-tailed test(1%) for 50 months.

Test1. Lattice, Monte-Carlo	1 P	2 P	3 P	5 P
put otm3	0	0	0	0
put otm2	0	0	0	3
put otm1	0	0	7	16
call otm1	0	0	10	24
call otm2	0	0	0	2
call otm3	1	1	3	0
Test2. Market, Monte-Carlo	1 P	2 P	3 P	5 P
put otm3	45	39	32	3
put otm2	41	34	20	6
put otm1	30	29	17	21
call otm1	23	20	18	24
call otm2	24	19	10	6
call otm3	13	12	21	1
Test3. Lattice, Market, Monte-Carlo	1 P	2 P	3 P	5 P
put otm3	45	39	32	3
put otm2	41	34	20	6
put otm1	30	29	17	21
call otm1	23	20	18	27
call otm2	24	19	10	6
call otm3	13	12	21	1

References

1) Black F. and Scholes M., The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 1973, 81, 637-654.
2) Dupire B., Pricing with a smile. *Risk*, 1994, January, 18-20.
3) Derman E. and Kani I., Riding on a smile. *Risk*, 1994, February, 32-39.
4) Rubinstein M., Implied Binomial Trees. *The Journal of Finance*, 1994, 49 , 771-818.
5) Li Y., "A new algorithm for constructing implied binomial trees: does the implied model fit any volatility smile?" *Journal of Computational Finance*, 2000/2001, 4, No. 2, 68-95.
6) Mawaribuchi J., Miyazaki K. and Okamoto M., 5-parameter Local Volatility Model: Fitting to Option Market Prices and Forecasting Ability. *IPSJ Transactions on Mathematical Modeling and Its Applications*, 2009, 2, No.2, 58-69. (in Japanese)
7) Hoshika H. and Miyazaki K., Which Deterministic Volatility Model Fits Market Price Exactly? Evidence from NIKKEI225 Option Market. *Journal of Japan Industrial Management Association*, 2008, 59, No.2, 451-458.