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不完全性定理再考

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不完全性定理が成り立たない算術体系（或いは，より広く形式体系）を作ることを目指して，まず，論理式の証明（決定）可能性と不可能性の根本的な違いを理解しようとして本研究を始めた．不完全性定理は，加算しか定義されていない算術でも，証明も反証もできない論理式があり，体系内では無矛盾性が証明できないと主張している．本文では次のことを明らかにした．①不完全性定理の証明に用いられた論理式は無限長である．②無限長論理式のゲーデル数は無限大であり，無限大のゲーデル数から元の論理式を復元できないから，ゲーデル数の概念（意図）は成功していない．③如何なる論理式でも，無限長論理式がある体系では不完全性定理が成り立つ．

Reflections on the incompleteness theorems

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As the first step to construct an arithmetic (or more generally, a formal system) without its incompleteness, we try to understand the provability (decidability) of a formula. The incompleteness thorems state that there is a formula which can not be proved and refuted and the consistency of the system can not be proved in the system. This paper descrives the following results; ① the Gödel number of the provability predicate to prove the incompleteness theorems is infinite, ② the intent of Gödel numbering is not fully realized, and ③ any formal system is incomplete, if it has formulas with infinite length. Finally, author's view on Hilbert's Consistency Program is stated.

1. Introduction

Gödel studied Peano arithmetic and proved that there is a predicate which we can not prove and not refute (the 1st incompleteness theorem), and the consistency of the arithmetic can not be proved in the arithmetic (the 2nd incompleteness theorem). It is believed that these results lead to the bankruptcy of Hilbert's Consistency Program. Under the influence of Gödel's paper Turing studied the halting problem of a computing machine which he proposed. The incompleteness theorems gave a great impact on information science.

The purpose of this research is understand the provability (decidability) of a formula. As the first step to approach to this problem we try to find the mechanism of the incompleteness. In this paper we show the followings. ① The length of any predicate which satisfies the diagonal theorem is infinite. Therefore, the length of the provability predicate is infinite. ② The intent of Gödel numbering is not fully realized. ③ Any formal system is incomplete, if it has formulas with infinite length. The author's view on Hilbert's Consistency Program is stated.

2. Preliminaries

We omite the descriptions of the propositional calculus, the predicate calculus, Robinson's Q, and Gödel numbering. The length of a formula is the number of the symbols in it. Gödel numbering satisfies the following characteristics.

(*1) Different finite sequences of expressions have different Gödel numbers.

(*2) There is an algorithm to decide whether a given number is the Gödel number of some finite sequence of expressions or not, and if it so, we can find the finite sequece of expressions.

Cosider the following two functions (formulas).

$$G_1(x) = S(x) \tag{1}$$

$$G_2(x) = f_1(x) \cdot S(1) + f_2(x) \cdot S(2) + \dots \tag{2}$$

where S is the successor function and $f_1(x)$ is a function such that if $x = 1$, $f_1(x) = 1$, and otherwise, $f_1(x) = 0$. $G_1(x)$ and $G_2(x)$ are functions with the same value, but their expressions are different. The Gödel number of $G_1(x)$ is finite and that of $G_2(x)$ is infinite. Gödel numbering is only based on expresions.

Let x be a free variable for natural numbers. Enumerate all functions with only x as a free variable in the ascending order of their Gödel numbers.

$$F_1(x), F_2(x), F_3(x), \dots \tag{3}$$

Consider the function which has the following values.

$$F_1(1), F_2(2), F_3(3), \dots \tag{4}$$

This function must exist in (3). Assume that the function is the qth function in (3), that is, $F_q(x)$. The function can be expressed in the following way.

$$F_q(x) = f_1(x) \cdot F_1(1) + f_2(x) \cdot F_2(2) + \dots \tag{5}$$

The function $G_2(x)$ with infinite length has another expression $G_1(x)$ with finite length. However $F_q(x)$ has not a finite length expression. If $h \neq k$, F_h and F_k are the different functions with different algorithms. Therefore, there are infinite different algorithms in $F_q(x)$. Then the length of $F_q(x)$ is infinite, and the Gödel number of $F_q(x)$ is infinite. The number of a formula whose length is infinite is infinite, and the Gödel number of each of them is infinite. We can not reconstruct the original formula from its Gödel number, if its length is infinite.

(*3) A formula with finite length satisfies (*1) and (*2).

(*4) A formula with infinite length does not satisfy (*1) and (*2).

Lemma 1
Gödel numbering is effective for formulas with finite length, but not for formulas with infinite length.

3. The diagonal theorem and its characteristics

The diagonal theorem is used to derive the provability predicate, that is, $\exists y Prov_T(x,y)$, where T is the set of axioms of logic and these of Q, x is the Gödel number of formula F to be proved, and y is the Gödel number of a proof of F. In this chapter we will show that the length of any formula in the diagonal theorem is infinite. First of all we will describe the outline of the proof of the diagonal theorem after Maehara' essay.
Enumerate all closed formulas in the ascending order of their Gödel numbers.

$$C_1, C_2, C_3, \ldots \tag{6}$$

$F_n(x)$ is the nth formula in (3). $F_n(n)$ which is obtained to substitute a natural number n for a free variable x, is a closed formula. Therefore, $F_n(n)$ must be in (6). Assume that it is $C_{\sigma(n)}$.

$$F_n(n) = C_{\sigma(n)} \tag{7}$$

【The diagonal theorem】
For any formula $F(x)$ that has only one free variable x, there exists a natural number p such that

$$F(p) \leftrightarrow C_p \tag{8}$$

The proof of this theorem is as follows. Consider $F(\sigma(x))$, where x is only one free variable in $F(\sigma(x))$. This formula can be find in (3) and assume that it is the qth formula in (3). That is

$$F(\sigma(x)) = F_q(x) \tag{9}$$

Substitute q for x. Then we have

$$F(\sigma(q)) = F_q(q) \tag{10}$$

From (7), we have

$$F_q(q) = C_{\sigma(q)} \tag{11}$$

Then the following equation is obtained.

$$F(\sigma(q)) = C_{\sigma(q)} \tag{12}$$

Putting $\sigma(q) = p$, we have the diagonal theorem.

Examine the proof. x in $F(\sigma(x))$ indicates the number(position) of a formula in (3), and $\sigma(x)$ is the number of a closed formula in (6). Since $\sigma(n)$ is determined by $F_n(n)$, the information about $F_n(n)$ is required to compute $F(\sigma(n))$. If $m \neq n$, $F_m(x)$ and $F_n(x)$ are different formulas. Then, those Gödel numbers are different. If we compute $F(\sigma(x))$ for $x = 1, 2, 3, \ldots$, we must compute infinite different closed logical fomulas. Therefore the Gödel number of $F(\sigma(x))$ is infinite.

Lemma 2
The length of the formula that satisfies the diagonal theorem is infinite.

Then, q in (9) is infinite. From (*4), $F(\sigma(x))$ does not satisfy (*1) and (*2). Since the length of $F(p)$ of (8) is infinite, we have Lemma 2.

Corollary
The length of the provability predicate $\exists y Prov_T(x, y)$ is infinite.

From Corollary and (*4), we have Lemma 3.

Lemma 3
We can not reconstruct the provability predicate from its Gödel number.

The concept of Gödel numbering is not effective in proving the incompleteness theorems. Therefore, the incompleteness theorems are not correctly proved.

4. Infinitely long formulas

Let S be a formal system in which all the rules are finite. By a proof in S, we mean a finite sequence of formulas, each of which either in an axiom or the conclusion of a rule whose hypotheses precede that formula in the proof. If A is the last formula in the proof P, P is a proof of A.

Evidently, any infinitely long formula can not be proved or refuted, because that a proof must be done in finite steps. Then we have the followings.

$$S \nvdash \varphi, \qquad S \nvdash \neg\varphi \tag{13}$$

(13) means the 1st incompleteness theorem. Furthermore we have

$$S \nvdash (\varphi \wedge \neg\varphi), \qquad S \nvdash \neg(\varphi \wedge \neg\varphi) \tag{14}$$

$\varphi \wedge \neg\varphi$ means the inconsistency of S, and $\neg(\varphi \wedge \neg\varphi)$ does the consistency of S.

We can prove neither the consistency nor the inconsistency of S. (14) means the 2nd incompleteness theorem.

Theorem 4
Any theories are incomplete and their consistency can not be proved, if they have some infinitely long formulas.

Note that theorem 4 holds for any formal theory. The provability predicate is an example of infinitely long formulas.

5. Concluding Remarks

[1] Gödel numbering does not work as Gödel intended. We must omit the description of it, if we explain the incompleteness of an arithmetic after Gödel. It is enough to write "Enumerate all functions with only one free variable x, that is, $F_1(x), F_2(x), F_3(x), \ldots$. The provability predicate exists definitely in it."

[2] It is believed that the incompleteness theorems destroy Hilbert's Consistency Program. Is it true ? Consider a destination which is infinitely far from here. Assume that we are moving with finite speed. It is self-evident that we can not reach the destination within finite hours. Someone says "This shows one of the limits of human's abilities". Yes, it is true. But, do you deeply impressed with his words. Do you find something important in his words. Definitely, no! The incompleteness theorems are similar to this situation. We have seen that any theories are incomplete and their consistency can not be proved, if they have some infinitely long formulas. Gödel's provability predicate is nothing but an example. Therefore, the incompleteness of a formal system is neither unusual nor mysterious. The concept of 'infinity' does not come from the real world, but it comes from ideal thinking. If we exclude 'infinity' from a formal system, this system could be complete. But further reserches are required. The problem is how to construct a formal

system. The author believe that Hilbert's Consistency Program is still alive.

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Appendix

We would like to describe the further findings of the research, though a part of them is under consideration. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be formulas. If the following three conditions are satisfied, the set $\{\varphi_1, \ldots, \varphi_n\}$ are called the covering formulas of φ.

(a) Each $\varphi_i (i = 1, \ldots, n)$ is the subformula of φ.

(b) For each pair φ_i and $\varphi_j (i \neq j)$, $\varphi_i(\varphi_i)$ is not a subformula of $\varphi_j(\varphi_i)$, respectively. That is, φ_i and φ_j are independent.

(c) The rest of deleting $\varphi_1, \ldots, \varphi_n$ are logical connectives and parentheses.

The followings are self-evident.

(1) If the length of φ is finite and each of the covering formulas of φ is proved or refuted, φ is proved or refuted.

(2) If the length of φ is infinite, φ is neither proved nor refuted.

Assume that the length of φ is finite. Note that if at least one of the covering formulas of φ is neither proved nor refuted, φ is neither proved nor refuted.

The basic symbols of the language of a formal system S is the set of variables, individual constants, function symbols and predicate symbols which define a language of S. Let f and P be a function and a predicate in the language, respectively. If $\forall x f(x)$ and/or $\exists x f(x)$ are in the mathematical axioms, $\forall x f(x)$ and/or $\exists x f(x)$ are added to the basic symbols. The same modification on the basic symbols is made concerning P.

Let $\varphi_1, \ldots, \varphi_n$ be the covering formulas of φ. Assume that $\varphi_k (1 \leq k \leq n)$ is neither proved nor refuted, and that the variables and individual constants in φ_k are the basic symbols of the language. If φ_k is expressed by the basic symbols and finite length, φ_k can be proved or refuted. Thefore, φ_k includes "defined functions" or "defined predicates" that are not in the basic symbols. If φ_k can be neither proved nor refuted, the length of the expressin of φ_k by the basic symbols must be infinite. From the above observations we have the following.

(3) Assume that the length of φ is finite. If φ expressed by the basic symbols is infinite, φ can be neither proved nor refuted.

Assume that S has a model, and each formula in S is expressed in the basic symbols. Let FL be the set of formulas with finite length, and φ be any formula in FL. Let $Cons = \neg(\varphi \wedge \neg\varphi)(= \varphi \vee \neg\varphi)$. We have the followings.

$$S \vdash \varphi, \quad or \quad S \vdash \neg\varphi \tag{15}$$

$$S \vdash Cons \tag{16}$$

To obtain a complete formal system we must exclude infinitely long formulas.